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Supersymmetric backgrounds from generalized Calabi-Yau manifolds

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Abstract: We show that the supersymmetry transformations for type II string theories on six-manifolds can be written as differential conditions on a pair of pure spinors, the exponentiated Kähler form e^{iJ} and the holomorphic form Ω. The equations are explicitly symmetric under exchange of the two pure spinors and a choice of even or odd-rank RR field. This is mirror symmetry for manifolds with torsion. Moreover, RR fluxes affect only one of the two equations: e^{iJ} is closed under the action of the twisted exterior derivative in IIA theory, and similarly Ω is closed in IIB. Modulo a different action of the B–field, this means that supersymmetric SU(3)-structure manifolds are all generalized Calabi-Yau manifolds, as defined by Hitchin. An equivalent, and somewhat more conventional, description is given as a set of relations between the components of intrinsic torsions modified by the NS flux and the Clifford products of RR fluxes with pure spinors, allowing for a classification of type II supersymmetric vacua on six-manifolds. We find in particular that supersymmetric six-manifolds are always complex for IIB backgrounds while they are twisted symplectic for IIA.

Keywords: Superstring Vacua, Supergravity Models, Differential and Algebraic Geometry.

Contents

1. Introduction

Compactifications with fluxes have received much attention recently due to a number of interesting features. In many ways these can be seen as extensions of the more conventional compactifications on Ricci-flat manifolds. On the other hand, many aspects of the latter, most notably in the case of Calabi-Yau manifolds, still have to find their generalized counterparts. Mirror symmetry has been one of the most prominent and useful features of Calabi-Yau compactifications, and the question of its extension to compactifications with fluxes is both of conceptual and of practical interest.

The issue of extending mirror symmetry to compactifications with fluxes has been studied recently in [1]–[4]. A first question is of course within which class of manifolds this symmetry should be defined. A natural proposal comes from the formalism of G-structures, recently used in many contexts of compactifications with fluxes. As shown in [3, 4], mirror symmetry can be defined on manifolds of SU(3) structure, thus generalizing the usual Calabi-Yau case. One of the points which makes this symmetry non–trivial is that, as expected, geometry and NS flux mix in the transformation. On the contrary, RR fluxes are mapped by mirror symmetry into RR fluxes and their transformation is well-understood. However, for many reasons it would be better to have a formalism that would incorporate geometrical data and fluxes in a natural way. This paper is a step in that direction. We will propose to use *pure spinors* as a formalism to describe SU(3)-structure compactifications.

The first reason to look for a unifying formalism is essentially checking the proposal for mirror symmetry given in [4]. In that paper, a quantitative rule for obtaining mirror-symmetric backgrounds is given based on the action of the twisted covariant derivative on spinors. From such rule one can read off the exchange of the components of the NS flux with the quantities describing the failure of the integrability of the complex structure and the Kähler form. As we will review in section 2, it works essentially exchanging representations $8+1 \leftrightarrow 6+\bar{3}$:

$$(\nabla J + H)_{ijk} \longleftrightarrow (\nabla J - H)_{i\bar{j}\bar{k}} \, .$$

Though checked on a number of examples, the formula is conjectural for the following reason: it was derived assuming that the manifold and the fluxes under consideration admit three Killing vectors, and then performing simultaneous T–duality along the three isometries. The same procedure is known in the context of Calabi-Yau mirror symmetry as the SYZ [5] approach. There, however, the structure of T^3 fibration was derived from considerations of moduli spaces of branes, which are lacking in compactifications with fluxes. However, the formula obtained under that assumption is clean enough to be conjectured to be valid when the T^3 fibration structure is not present.

Inclusion of RR fluxes gives a check of the conjecture in the following sense. Mirror symmetric compactifications should yield the same physics in four dimensions. In particular, a compactification which preserves supersymmetry should be sent to another one with the same property. Since on supersymmetric backgrounds the total NS and RR contributions to the supersymmetry equations sum up to zero, demanding that mirror symmetry maps supersymmetric backgrounds to supersymmetric backgrounds allows to check if the proposed NS transformation is compatible with the known RR one and if so to lend further support for the whole picture.

It is easy to see that this check has a chance of working realizing that two objects are the same: Clifford$(6,6)$ *spinors* and *bispinors*. These appear in NS and RR sector respectively. The Clifford$(6,6)$ spinors appeared in [4] in order to interpret the mirror symmetry formulae in a more natural way. As far as we are concerned in this introduction, Clifford$(6,6)$ spinors are simply *formal sums of forms*. (In analogy with usual spinors, which are often realized as formal sums of $(0,p)$ forms.) Such a spinor is called *pure* if it is annihilated by half of the gamma matrices. A pure spinor defines an SU$(3,3)$ structure on the bundle $T+T^*$ on the manifold. If the spinor is also closed, the manifold is called by Hitchin [6] a *generalized Calabi-Yau.*[1] For a SU(3) structure on T, there are two pure spinors, which are orthogonal and of unit norm. An SU(3) structure is defined by a two-form J and a three-form Ω obeying $J \wedge \Omega = 0$ and $i\Omega \wedge \bar{\Omega} = (2J)^3/3!$. Then, the two pure spinors are e^{iJ} and Ω. From this point of view it is natural to conjecture that mirror symmetry between two SU(3) structure manifolds exchanges these two pure spinors.

It is also possible to incorporate the B-field since multiplying a pure spinor by e^B leaves it pure [6]. This is indeed what happened in [4]: T-duality along T^3 (when it is

[1] In [7], the same name is used for a different type of manifold, that has two pure spinors whose associated generalized complex structures are commuting and integrable.

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possible) realizes the exchange

$$e^{B+iJ} \longleftrightarrow \Omega\,,$$

thus motivating the introduction of the Hitchin formalism just mentioned. In the Calabi-Yau case, this exchange is implicit in many applications of mirror symmetry.

The second fact, that RR fields are described by bispinors, is much more standard and familiar from the very spectrum of the superstring. In this paper we will use in many instances that the Clifford(6,6) spinors above can be identified with bispinors (Clifford(6)$\times$ Clifford(6)) under the map from forms to elements of the Clifford(6) algebra, $dx^m \to \gamma^m$.

Using this identification, we will be able to show that the supersymmetry transformations for IIA and IIB can be written in a unified fashion using formally two pure spinors and a total RR field of either even or odd rank. Very schematically

$$\delta\psi_m = [D_m^H + (\varphi_1 \cdot F)_m + (\varphi_2 \cdot F)_{mn}\gamma^n]\epsilon\,, \tag{1.1}$$

F here is the formal sum of all RR fields, the dot stands for a Clifford multiplication and φ_1, φ_2 are pure spinors. It is not hard to see that choosing the RR field F to be even or odd fixes the roles played by each pure spinor which has to be even or odd as well. Mirror symmetry then will exchange the pure spinors and change the rank of the RR field from odd to even and vice versa. Essentially formulae as (1.1) come from defining SU(3) structures in terms of an *ordinary* spinor η, and then using Fierz identities to express the two pure spinors as $e^{iJ} = 8\eta_- \otimes \eta_-^\dagger$, $\Omega = 8i\eta_+ \otimes \eta_-^\dagger$. In these terms, mirror symmetry can be seen as conjugation on a sector.

In fact, there is a more precise sense in which the formalism of pure spinors is relevant. Indeed, in section 5 we will see that the supersymmetry equations imply differential equations for the pure spinors, schematically

$$\begin{aligned} e^{-f_1} d(e^{f_1}\varphi_1) &= H \bullet \varphi_1 \\ e^{-f_2} d(e^{f_2}\varphi_2) &= H \bullet \varphi_2 + (F, \varphi)\,. \end{aligned} \tag{1.2}$$

The operator $H\bullet$ is a certain action of the H three-form, involving contractions and wedges but *different* from $H\wedge$. So, in both IIA and IIB there is a "preferred" pure spinor (of the same parity as F - namely e^{iJ} in IIA and Ω in IIB) which does not receive any back reaction from the RR fluxes. This property is called (twisted) generalized Calabi-Yau [6]. The twisting refers to the presence of the H field. In the mathematical literature (and in some physical applications [8]) this twisting is actually always appearing in the form $(d + H\wedge)$. It is interesting to see that in general the inclusion of RR fluxes requires a different form of twisting than the one usually assumed. Understanding the origin of this twisting from first principles would be of some importance.

Much of this discussion can be carried out in more conventional (and also somewhat more practical from point of view of finding examples) terms. In supersymmetry transformations one can well separate the NS and RR contributions. The former are given by components of SU(3) intrinsic torsion modified by inclusion of the NS three-form flux. The latter are Clifford products of RR fluxes with geometric data (pure spinors again) consistent with (1.1). It turns out that the RR fluxes affect only some of the components of the intrinsic torsion (compare to (1.2)), thus making the analysis of supersymmetry conditions rather easy and allowing for a complete classification of type II theories on six-manifolds. In particular we show that the supersymmetric geometries in IIB are always complex while in IIA they are twisted symplectic. Mirror symmetry can also be seen as well as the respective mappings of RR-corrected and RR-uncorrected sectors in IIA and IIB (in a agreement with $6+\bar{3} \leftrightarrow 8+1$ rule).

This two-level discussion ("spinors" versus "pure spinors") is reflected in the structure of the paper, which has two complementary but self-contained parts. In section 2 we review the basics of the formalism and the way mirror symmetry works for geometry and B-field. We proceed to describe the general features of RR supergravity transformations in section 3, where we also show how these can be put in a manifestly mirror symmetric way on manifolds of SU(3) structure. Analysis of the supersymmetry conditions is presented in section 4. Finally, in section 5 we discuss these conditions in terms of pure spinors and show in particular the correspondence between supersymmetric string vacua and generalized Calabi-Yau manifolds.

2. NS flux and geometry

In this section we briefly review the action of mirror symmetry on the NS sector, to set the stage but also to clarify some points from [4].[2]

We start by briefly introducing the notions of SU(3)-structure and intrinsic torsion with the help of which we will describe the non-Ricci-flat geometries under consideration. A manifold with SU(3)-structure has all the group-theoretical features of a Calabi-Yau, namely invariant two- and three forms, J and Ω respectively. On a manifold of SU(3) *holonomy*, not only J and Ω are well defined, but they are also closed: $dJ = 0 = d\Omega$. If they are not closed, dJ and $d\Omega$ give a good measure of how far the manifold is from having SU(3) holonomy

$$dJ = -\frac{3}{2}\,\mathrm{Im}(W_1\bar{\Omega}) + W_4 \wedge J + W_3\,, \tag{2.1}$$

$$d\Omega = W_1 J^2 + W_2 \wedge J + \bar{W}_5 \wedge \Omega\,. \tag{2.2}$$

The W's are the $(3\oplus\bar{3}\oplus 1)\otimes(3\oplus\bar{3})$ components of the intrinsic torsion: W_1 is a complex zero-form in $1\oplus 1$, W_2 is a complex primitive two-form, so it lies in $8\oplus 8$, W_3 is a real primitive $(2,1)\oplus(1,2)$ form and it lies in $6\oplus\bar{6}$, W_4 is a real one-form in $3\oplus\bar{3}$, and finally W_5 is a complex $(1,0)$-form (notice that in (2.2) the $(0,1)$ part drops out), so its degrees of freedom are again $3\oplus\bar{3}$.

These W_i allow to classify the differential type of any SU(3) structure. A simple counting tells that there can be a representation of the intrinsic torsion given simply by a six-by-six matrix and a vector. There is indeed an alternative definition of the torsions which

[2] The discussion here is far from being complete and is concerned mostly with the spinorial aspects of the formalism. We refer to [3, 4] and references therein for a more complete account of G-structures and intrinsic torsions.

essentially does this. A SU(3) structure can be defined also by a spinor η. In terms of this, J and Ω above are defined as bilinears: $\eta^\dagger\gamma_{mn}\gamma\,\eta = iJ_{mn}$ and $-i\eta^\dagger\gamma_{mnp}(1+\gamma)\eta = \Omega_{mnp}$, where γ is the six dimensional chirality operator.

The spinor η also gives a basis for all spinors on the manifold: η, $\gamma\eta$ and $\gamma^m\eta$.[3] Anything else in the Clifford algebra acting on η, say $\gamma^{m_1\ldots m_n}$, can be re-expressed in terms of this basis. So, in general we can write[4]

$$D_m\eta = (\tilde{q}_m + iq_m\gamma + iq_{mn}\gamma^n)\,\eta\,. \tag{2.3}$$

The q's, defined by this equation, are real, and provide just another definition of the intrinsic torsion. It is immediate to notice that there is a certain redundancy in (2.3): it has three vectors (q_m, $\tilde{q}_m$ and one from q_{mn}), and this constrained trio is the counterpart of the more economical pair given by W_4 and W_5. There are two natural ways of resolving this ambiguity. One, which was used in [4], consists in noticing that only the $(1,0)$ part of $\tilde{q}_m + iq_m$ appears in W_4, W_5, and consequently assuming it has no $(0,1)$ part. Here we will use another method. Indeed, normalizing the spinor η to have *constant* norm allows to set $\tilde{q}_m = 0$.

Having fixed the ambiguity, q and W are simply related by a change of basis, which in holomorphic/antiholomorphic indices reads

$$\begin{aligned} q_{ij} &= -\frac{i}{8}W^3_{ij} - \frac{1}{8}\Omega_{ijk}\bar{W}^k_4\,, \\ q_{i\bar{\jmath}} &= -\frac{i}{4}\bar{W}^2_{i\bar{\jmath}} + \frac{1}{4}\bar{W}_1 g_{i\bar{\jmath}}\,, \qquad q_i = \frac{i}{2}(W_5 - W_4)_i\,. \end{aligned} \tag{2.4}$$

Here $W_i = W_i^+ - iW_i^-$ as usual in the literature, and we have defined $W^3_{mn} = W^3_{mpq}\Omega^{pq}{}_n$

The q's can also be related to the covariant derivatives of J and Ω. Let us define the real and imaginary parts of Ω as $\Omega = \psi - i\tilde{\psi}$. In real indices, then, we have

$$q_{mn} = \frac{i}{8}\nabla_m J_{pq}\tilde{\psi}^{pq}{}_n\,. \tag{2.5}$$

Also, one can directly think in terms of the so-called contorsion κ_{mnp}, defined as $\nabla_m\eta = \kappa_{mnp}\gamma^{np}\eta$:

$$\kappa_{mnp} = \frac{i}{8}\nabla_m J_{ns}J^s{}_p + \frac{1}{288}(\tilde{\psi}^{rst}\nabla_m\psi_{rst})J_{np}\,.$$

The basis (2.2) is usually more popular because easier to analyze. For instance, looking at it one immediately concludes that $W_1 = W_2 = 0$ iff the manifold is complex. Indeed the $(2,2)$ part of $d\Omega$, $W_1J^2 + W_2\wedge J$, would be absent in the case of integrable complex structure.

[3] In the following, differently from notation in [4], we will denote real six-dimensional indices as $m, n, \ldots$ and holomorphic/antiholomorphic indices as $i, j, \ldots$ $(\bar{\imath}, \bar{\jmath}, \ldots)$.

[4] The notation for q_m and $\tilde{q}_m$ is the opposite as that used in [4], since shortly we will set $\tilde{q}$ to zero by normalizing the spinor and keep q_m nonzero.

From other side, the spinorial approach treats J and Ω more symmetrically and turns out to be much more T-duality friendly. It is indeed immediate to notice that there is no natural exchange of the quantities W in (2.2); it was this fact that prompted the definition of (2.3).

We are ready now to discuss the T-duality/mirror transformation.[5] For a generic metric (with a nontrivial connection on the fiber) neither W's nor q's can be invariant under T-duality - they necessarily mix with the flux

$$H = -\frac{3}{2}\,\mathrm{Im}(H^{(1)}\bar{\Omega}) + H^{(3)} \wedge J + H^{(6)}\,, \tag{2.6}$$

where the flux components are labeled according to the representation, namely $H^{(1)}$ is the $1\oplus 1$ complex scalar, $H^{(6)}$ the $6\oplus\bar{6}$ real 3-form and $H^{(3)}$ the $3\oplus\bar{3}$ real 1-form. Including the flux will lead to a complexification of the components of the intrinsic torsion in matching representations; however, nontrivial transformations must mix different representations. This is for the following reason. The two mirrors have two different SU(3) structures since the two SU(3) are differently embedded into Spin(6,6), because the fiber directions change from the tangent bundle to the cotangent bundle. As a result, representations get actually mixed. Indeed, as we noticed already, the W in (2.2) have no natural "pairing".

There are two ways one goes around this problem. One is to go to the base, where one can define T-duality invariants, and further decompose the SU(3) representations thus allowing them to mix. Alternatively, we could consider the sum of the tangent and cotangent bundles and take sums of representations. Our final formula for mirror symmetry will be doing exactly this.

The first approach is the one proven in [4] using T-duality. Assuming that the initial six-dimensional manifold has a fibration structure, one can decompose the torsions under the SO(3) of the base, which is smaller of the total SU(3). W_2 and W_3 get then split as $W_2 = w_2^s + w_2^a$ $(8 \to 5 \oplus 3)$ and $W_3 = w_3^s + w_3^t$ $(6 \to 5 \oplus 1)$. Using the usual expressions for the metric and the B-field in the torus-fibered case, we get under T-duality

$$\begin{aligned} W_1 - iH_{(1)} &\longleftrightarrow -\overline{(W_1 - iH_{(1)})}\,, \\ \bar{w}_2^s &\longleftrightarrow w_3^s - iH^{(6)s}\,, \\ W_5, \bar{w}_2^a &\longleftrightarrow W_4 - iH^{(3)}\,. \end{aligned} \tag{2.7}$$

T-duality preserves SO(3) representations, as it should since it does not act on the base. More interesting is to notice that a complexification occurred naturally between certain components of W and flux H.

Thus one can add in (2.3) a dependence on H to the covariant derivative (and as a consequence the intrinsic torsion)

$$D^H_m \eta = i\,(Q_m \gamma + Q_{mn}\gamma^n)\,\eta\,. \tag{2.8}$$

The idea is to construct D^H in such a way as to find good T-duality transformation properties afterwards. Not too surprisingly, one finds that the best definition is exactly the

[5] The transformation rules for the fields as well as the working assumptions can be found in [4], here we just quote some relevant results.

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same as that in the supergravity supersymmetry transformations: $D^H_m \equiv (D_m + \frac{1}{8} H_{mnp}\gamma^{np})$. The "twisted" components of the intrinsic torsion turn out to be diagonal under T-duality: elements of the basis transform picking a sign. In holomorphic/antiholomorphic indices,

$$\begin{aligned} Q_{ij} &= -\frac{i}{8}\left(W_3 + iH^{(6)}\right)_{ij} - \frac{1}{8}\Omega_{ijk}\left(\bar W_4 + i\bar H^{(3)}\right)^k , \\ Q_{i\bar\jmath} &= -\frac{i}{4}\bar W^2_{i\bar\jmath} - \frac{1}{4}\left(\bar W_1 + 3i\bar H^{(1)}\right) g_{i\bar\jmath} , \qquad Q_i = \frac{i}{2}(W_5 - W_4 - iH^{(3)}) . \end{aligned} \tag{2.9}$$

So, adding H as $D \to D^H$ complexifies W as $W + iH$. Using (2.7), one can verify that this "complexified" Q's, when restricted to the base, transform nicely, essentially picking $\pm$. So, the spinorial basis is more suited for T-duality than the original W basis.

The last step is now to conjecture a transformation rule which might be valid also in cases which are *not* T^3 fibrations. One guideline is the transformation rule found above. Another one is that, as mentioned in the introduction, mirror symmetry sends supersymmetric vacua to supersymmetric vacua. In a sense, T-duality is induced by an exchange of ϵ_+ with ϵ_-. Since we also have $\gamma^{\bar\imath}\epsilon_- = 0$, one arrives at

$$Q_{ij} \longleftrightarrow -Q_{i\bar\jmath} , \qquad Q_i \longleftrightarrow -\bar Q_i . \tag{2.10}$$

This is the other way of getting around the lack of natural pairing of representations in (2.2). Qualitatively we have $6 + \bar 3 \leftrightarrow 8 + 1$.

We can also express this exchange in a way which maybe is more mnemonical. Using (2.5) we have

$$(\nabla J + H)_{ijk} \longleftrightarrow (\nabla J - H)_{i\bar\jmath\bar k} ; \tag{2.11}$$

notice that $\nabla_i J_{\bar\jmath\bar k}$ is automatically zero by hermiticity of J ($J_{i\bar\jmath} = i g_{i\bar\jmath}$) and the fact that the connection is Levi-Civita. In fact, in this form the symmetry $6 + \bar 3 \leftrightarrow 8 + 1$ was already noticed long ago by Salamon [9] in a different context predating the mirror symmetry.[6] On a manifold M of any dimension, one can consider the bundle $\mathcal{T}$ of almost complex structures, or, equivalently, of lines of pure spinors: this is called *twistor* bundle. In [9] it was shown how to define two almost complex structures F_i on the total space of $\mathcal{T}$. There are relations between the behaviour of these two almost complex structures F_i and our split q_{ij} versus $q_{i\bar\jmath}$ above. For example, a section of the twistor bundle is a holomorphic submanifold with respect to the almost complex structure F_1 if $q_{i\bar\jmath} = 0$, whereas it is holomorphic with respect to F_2 if $q_{ij} = 0$. (Note that neither (2.11) nor the pair of the complex structures on the twistor bundle refer to Ω and therefore do not involve W_5. Thus some of the arguments here may be extended to manifolds where the structure group is given by U(3) rather than SU(3).) These results seem clearly to be relevant to a further understanding of mirror symmetry, and it would be nice to realize them physically. Maybe a model on the twistor space is the right way to prove mirror symmetry from first principles.

We will conclude this section with a brief review of the conditions for supersymmetry in the case with H-flux only [10, 11], which in this language become conditions for the

[6]We thank S. Salamon for pointing out his work and for useful discussions on this section.

vanishing of components of Q's. In many ways this example sets the stage for our discussion in a sense that it gives two basic equations for the pure spinors, which then may or may not be modified by the RR back reaction.

To have supersymmetry it is enough that one chirality, say η_+, is annihilated by D^H. From (2.3) we have

$$\begin{aligned} D^H_i \eta_+ &= iQ_i\eta_+ + iQ_{ij}\gamma^j\eta_- \,, \\ D^H_{\bar\imath} \eta_+ &= iQ_{\bar\imath}\eta_+ + iQ_{\bar\imath j}\gamma^j\eta_- \,. \end{aligned} \tag{2.12}$$

Here η is also normalized to one; as we said, the spinors preserved by supersymmetry often do not have this property, but we can always rescale them. Notice that $Q_{i\bar\jmath}$ and $Q_{\bar\imath\bar\jmath}$ have disappeared from $D^H\eta_+$, because η_-, being a Clifford vacuum, is annihilated by $\gamma^{\bar\imath}$. From (2.12) it follows directly that the complexified Q_{ij} and $Q_{\bar\imath j}$ have to vanish. These will say that the complexified W_3 has to be purely antiholomorphic, which in more usual terms means of type $(1,2)$ (this is the condition $W_3 = *H_3$) and that W_2 has to vanish. The vectors require a little more care because usually the dilaton is rescaled in the metric (as a warping) and in the spinor itself. More generally it is clear that one can use the gamma matrices identities mentioned above to reduce the expression to a form like (2.3), and then use (2.4).

In this case, supersymmetry is trivially consistent with the proposal for mirror symmetry (2.10), since both Q_{ij} and $Q_{i\bar\jmath}$ are zero. In this form the duality might seem a little tautological, in the sense that it sends a supersymmetric vacuum in another one in an obvious way. Compare however with the usual mirror symmetry: a Calabi-Yau is sent to another Calabi-Yau, and the non triviality lies in the exchange of Kähler and complex structure moduli. A better understanding of the moduli in the flux compactifications is one important open question.

Another way to see that the conditions for this case are consistent with the mirror symmetry proposal is to put them in a manifest symmetric way

$$\left(D_m + \frac{1}{4}H_{mnp}dx^n\iota^p\right)e^{iJ} = 0\,, \qquad \left(D_m + \frac{1}{4}H_{mnp}dx^n\iota^p\right)\Omega = 0\,, \tag{2.13}$$

where $\iota^m \equiv g^{mn}\iota_{\partial_n}$.[7] D_m can be written as a covariant derivative for bispinors or as the usual Levi-Civita covariant derivative on forms. The object e^{iJ} is a formal sum of forms; its meaning will be explained in section 5. In section 5 we will also see how (2.13) get modified in presence of RR fields.

3. RR fluxes

In this section we consider the introduction of RR fluxes and analyze how this affects the supersymmetry conditions and mirror symmetry. The idea is to generalize what done in the previous section for the NS fluxes. Just as the entire NS contribution to the covariant derivative of the invariant spinor was summarized in Q's (see (2.9)), the RR contribution

[7] $\iota_{\partial_n}\colon \Lambda^p T^* \to \Lambda^{p-1}T^*$, $\iota_{\partial_n} dx^{i_1}\wedge\cdots\wedge dx^{i_p} = p\delta_n^{[i_1}dx^{i_2}\wedge\cdots\wedge dx^{i_p]}$.

can be accounted for by the introduction of similar objects, R_m and R_{mn}, with a group decomposition matching that of Q's. The condition for supersymmetric vacua will then reduce to algebraic equations of the form $R = -Q$. This formulation is also more suitable to check the mirror symmetry proposal.

To write the analogue of (2.9) for non-zero RR fluxes we need to discuss in more detail the supersymmetry transformations for type II theories and the general features of the solutions we are looking for.

Our starting point is the democratic formalism of [12], for which the supersymmetry transformations for the gravitino and the dilatino are[8]

$$\delta\psi_M = D_M\epsilon + \frac{1}{4}H_M\mathcal{P}\epsilon + \frac{1}{16}e^{\phi}\sum_n \not{F}_{2n}\,\Gamma_M\mathcal{P}_n\,\epsilon\,, \tag{3.1}$$

$$\delta\lambda = \left(\not{\partial}\phi + \frac{1}{2}\not{H}\mathcal{P}\right)\epsilon + \frac{1}{8}e^{\phi}\sum_n(-1)^{2n}(5-2n)\not{F}_{2n}\,\mathcal{P}_n\epsilon\,. \tag{3.2}$$

$F_{2n} = dC_{2n-1} - H\wedge C_{2n-3}$ are the modified RR field strengths with non standard Bianchi identities, that we will call from now on simply RR field strengths, $n = 0,\ldots,5$ for IIA and $n = 1/2,\ldots,9/2$ for IIB and $H_M \equiv \frac{1}{2}H_{MNP}\Gamma^{NP}$.

Note that the "total" RR field involves both the field strengths and their duals, and a self-duality relation is still to be imposed

$$F_{2n} = (-1)^{Int[n]}\star_{10} F_{10-2n}\,. \tag{3.3}$$

The definitions of $\mathcal{P}$ and $\mathcal{P}_n$ are different in IIA and IIB: for IIA $\mathcal{P} = \Gamma_{11}$ and $\mathcal{P}_n = \Gamma_{11}^n\sigma^1$, while for IIB $\mathcal{P} = -\sigma^3$, $\mathcal{P}_n = \sigma^1$ for $n+1/2$ even and $\mathcal{P}_n = i\sigma^2$ for $n+1/2$ odd. The two Majorana-Weyl supersymmetry parameters of type II supergravity are arranged in the doublet $\epsilon = (\epsilon_1,\epsilon_2)$.

It is useful to note that in the combination $\Gamma_M\delta\psi_M - \delta\lambda$ the term corresponding to RR fluxes cancels. So instead of using the gravitino and the dilatino equations, we will work with the gravitino and the modified dilatino equation

$$\Gamma^M\delta\psi_M - \delta\lambda = \left(\not{D} - \not{\partial}\phi + \frac{1}{4}\not{H}\mathcal{P}\right)\epsilon\,. \tag{3.4}$$

We are interested in solutions corresponding to warped compactifications to 4d Minkowski space-time. So the 10d metric can be written as

$$ds^2_{10} = e^{2A(y)}dx_\mu dx^\mu + ds^2_6(y)\,, \tag{3.5}$$

and we decompose gamma matrices, spinors and forms into 4d and 6d parts.

We choose a Majorana representation for the 10d gamma matrices and we split them according to

$$\Gamma_\mu = \gamma_\mu\otimes 1\,,\qquad \Gamma_m = \gamma_5\otimes\gamma_m\,, \tag{3.6}$$

[8]Our convention for $\not{F}^{(2n)}$, differs from that in [12] in that we include a factor of $1/(2n)!$ in the definition of the slash (cf. eq. (5.1)).

where the 6d gammas are anti-hermitean and purely imaginary, and

$$\gamma_5 = i\frac{1}{4!}\epsilon_{\mu\nu\lambda\rho}\gamma^{\mu\nu\lambda\rho}\,, \qquad \gamma_7 = -i\frac{1}{6!}\epsilon_{mnpqrs}\gamma^{mnpqrs}\,. \tag{3.7}$$

With the above choice for the 10d gamma matrices, we can now consider the decomposition of the spinors ϵ_i. In Type IIA the two supersymmetry parameters have opposite chirality, i.e.

$$\gamma_{11}\epsilon_1 = \epsilon_1\,, \qquad \gamma_{11}\epsilon_2 = -\epsilon_2\,, \tag{3.8}$$

and can be decomposed as

$$\begin{aligned} \epsilon_1 &= \zeta_+ \otimes \eta^1_+ + \zeta_- \otimes \eta^1_-\,, \\ \epsilon_2 &= \zeta_+ \otimes \eta^2_- + \zeta_- \otimes \eta^2_+\,, \end{aligned} \tag{3.9}$$

where ζ and η^i are chiral spinors in 4 and 6 dimensions, respectively. The Majorana condition implies also $(\zeta_+)^* = \zeta_-$, $(\eta^i_+)^* = \eta^i_-$.

For IIB, the two ϵ have the same chirality and we choose it to be positive, which leads to the decomposition

$$\epsilon_i = \zeta_+ \otimes \eta^i_+ + \zeta_- \otimes \eta^i_-\,, \tag{3.10}$$

where again $\zeta^*_+ = \zeta_-$, $\eta^{i*}_+ = \eta^i_-$.

Finally, we need to decompose the RR field strengths. In order to preserve 4d Poincare invariance, they should be of the form

$$F_{2n} = \hat{F}_{2n} + Vol_4 \wedge \tilde{F}_{2n-4}\,. \tag{3.11}$$

Here $\hat{F}_{2n}$ stands for purely internal fluxes. The self-duality of F_{2n} now becomes $\tilde{F}_{2n-4} = (-1)^{Int[n]} \star_6 \hat{F}_{10-2n}$, and allows to write the RR part of (3.1) in terms of the internal fluxes only. From now on we will work with only internal fluxes, and drop the hats in F.

With the above decompositions, the supersymmetry conditions (3.1), (3.4) reduce to a set of equations on the two spinors η^i_+. Having two internal spinors would give a SU(2) structure. In this paper we are interested in manifolds with SU(3) structure, which is defined by a single spinor η_+. Then we should find a way to relate the spinors η^1_+ and η^2_+ to the spinor η_+. If η_+ is normalized ($\eta^\dagger_+\eta_+ = \frac{1}{2}$), the most general way the η^i_+ and η_+ can be related is

$$\eta^1_+ = a\eta_+\,, \qquad \eta^2_+ = b\eta_+\,, \tag{3.12}$$

where a and b are complex functions of the internal space. Similarly, complex conjugate relations hold for the negative chirality spinors. In order to be able to define the RR analogues of the Q's, we must then express all the spinors on the internal manifold in terms of the basis

$$\eta_\pm,\ \gamma^m\eta_\pm\,. \tag{3.13}$$

Coming back to the supersymmetry conditions for IIA, they become

$$\alpha \not\partial A\eta_+ + \frac{i}{4}e^\phi \not F_{A1}\eta_- = 0\,, \tag{3.14}$$

$$\alpha D_m\eta_+ + (\partial_m\alpha + \frac{1}{4}\beta \not H_m)\eta_+ + \frac{i}{8}e^\phi \not F_{A1}\gamma_m\eta_- = 0\,, \tag{3.15}$$

$$\alpha \not D\eta_+ + \left[\alpha\not\partial(2A - \phi + log\alpha) + \frac{1}{4}\beta \not H\right]\eta_+ = 0\,, \tag{3.16}$$

while for IIB they are

$$\left[\alpha \partial\!\!\!/ A + \frac{i}{4} e^{\phi} F\!\!\!/_{B1}\right] \eta_+ = 0\,, \tag{3.17}$$

$$\alpha D_m \eta_+ + \left[\partial_m \alpha - \frac{1}{4}\beta H\!\!\!/_m - \frac{i}{8} e^{\phi} F\!\!\!/_{B1} \gamma_m\right] \eta_+ = 0\,, \tag{3.18}$$

$$\alpha D\!\!\!\!/\, \eta_+ + \left[\alpha \partial\!\!\!/ (2A - \phi + log \alpha) - \frac{1}{4}\beta H\!\!\!/\right] \eta_+ = 0\,, \tag{3.19}$$

where we have introduced $\alpha = a + ib$ and $\beta = a - ib$.

In both cases, the first and second equations come from the spacetime and the internal gravitino respectively, while the third one comes from the modified dilatino. The RR field strengths have been grouped in the following way:

$$\begin{aligned} -F_{A1} &\equiv \beta^* F_0 + \alpha^* F_2 + \beta^* F_4 + \alpha^* F_6\,, \\ F_{A2} &\equiv \alpha^* F_0 + \beta^* F_2 + \alpha^* F_4 + \beta^* F_6\,, \end{aligned} \tag{3.20}$$

$$\begin{aligned} F_{B1} &\equiv \alpha F_1 - \beta F_3 + \alpha F_5\,, \\ -F_{B2} &\equiv \beta F_1 - \alpha F_3 + \beta F_5\,. \end{aligned} \tag{3.21}$$

A second set of equations with $F_{A1} \to F_{A2}$, $F_{B1} \to F_{B2}$ and $\alpha \leftrightarrow \beta$ comes from the supersymmetry transformations of the second gravitino and dilatino.

As for the NS case, the idea now is to reduce all the terms appearing in the supersymmetry variations to the spinor basis (3.13), by decomposing the action of a generic gamma matrix $\gamma^{m_1,\ldots,m_n}$ on $\eta_\pm$. This allows to treat the supersymmetry constraints for IIA and IIB on a common ground. The generic form of the equations is now

$$\begin{aligned} \delta\Psi_m :&\ \ i(Q_m + R_m)\eta_+ + i(Q_{mn} + R_{mn})\gamma^n \eta_- = 0\,, \\ \delta\Psi_\mu :&\ \ S\eta_- + (S_m + A_m)\gamma^m \eta_+ = 0\,, \\ \delta\lambda :&\ \ T\eta_- + T_m \gamma^m \eta_+ = 0\,, \end{aligned} \tag{3.22}$$

where Q's, T's and A's contain the contribution from the geometry and the NS fluxes, and the S's and R's come from the RR fluxes. This general structure holds for both IIA and IIB, even though the explicit form of Q, S and R is theory dependent. The actual computation of these coefficients can be found in the appendix. Here we will simply list the results. For IIA we have

$$\begin{aligned} A_m &= \alpha \partial_m A\,, \\ S &= \frac{i}{4} e^{\phi} (F\!\!\!/_{A1} e^{iJ\!\!\!/})_0\,, \\ S_m &= \frac{1}{4} e^{\phi} \mathrm{Re}\Big[(F\!\!\!/_{A1} \bar{\Omega\!\!\!/})_m\Big]\,, \\ Q_m &= -i\partial_m \alpha + \frac{1}{2} J_m{}^n \left(\alpha W_5 - \alpha W_4\right)_n + \frac{1}{2}\beta H^{(3)}_m\,, \end{aligned}$$

$$
\begin{aligned}
Q_{mn} &= \mathrm{Re}\Big[\frac{1}{2}(\alpha W_1 + 3i\beta H^{(1)})\bar{P}_{mn} - \frac{1}{4}\Omega_{mnp}(\alpha W_4 + i\beta H^{(3)})^p - \\
&\qquad - \frac{i}{8}(\alpha W_3 + i\beta H^{(6)})_{mn} + \frac{i}{2}\bar{P}_m{}^p \alpha W_{2\;pn}\Big] , \\
R_m &= -\frac{i}{8} e^{\phi} (\overline{\not{\Omega}} \not{F}_{A1})_m , \\
R_{mn} &= \frac{1}{4} e^{\phi} \mathrm{Re}\left[-(\not{F}_{A1\,m} e^{i\not{J}})_n + \frac{1}{2}(\not{F}_{A1} e^{i\not{J}})_0 g_{mn} + (\not{F}_{A1} e^{i\not{J}})_{mn}\right] , \\
T &= \frac{3}{2}\left(i\alpha W_1 - \beta H^{(1)}\right) , \\
T_m &= \alpha\, \partial_m (2A - \phi + \log\alpha) + \alpha \left[W_{4\,m} + \frac{i}{2} J_m{}^n (W_5 - W_4)_n\right] - \frac{1}{2} J_{mn}\beta H^{(3)}_n ,
\end{aligned}
\tag{3.23}
$$

where we have defined $\not{F}_m \equiv \frac{1}{k!} F_{m i_1 \ldots i_k}\gamma^{i_1\ldots i_k}$. Similarly for IIB we get

$$
\begin{aligned}
A_m &= \alpha \partial_m A , \\
S &= \frac{1}{4} e^{\phi} (\not{F}_{B1} \not{\Omega})_0 , \\
S_m &= \frac{1}{4} e^{\phi}\, \mathrm{Re}\left[(\not{F}_{B1} e^{-i\not{J}})_m \right] , \\
Q_m &= -i\partial_m \alpha + \frac{1}{2} J_m{}^n\, (\alpha W_5 - \alpha W_4)_n - \frac{1}{2}\beta H^{(3)}_m , \\
Q_{mn} &= \mathrm{Re}\Big[\frac{1}{2}(\alpha W_1 - 3i\beta H^{(1)})\bar{P}_{mn} - \frac{1}{4}\Omega_{mnp}(\alpha W_4 - i\beta H^{(3)})^p \; - \\
&\qquad - \frac{i}{8}(\alpha W_3 - i\beta H^{(6)})_{mn} + \frac{i}{2}\bar{P}_m{}^p \alpha W_{2\;pn}\Big] , \\
R_m &= -\frac{1}{8} e^{\phi} (e^{-i\not{J}} \not{F}_{B1})_m , \\
R_{mn} &= \frac{1}{4} e^{\phi} \mathrm{Re}\left[i(\not{F}_{B1\,m}\not{\Omega})_n - i(\not{F}_{B1}\not{\Omega})_{mn} - i\frac{1}{2}(\not{F}_{B1}\not{\Omega})_0\, g_{mn}\right] , \\
T &= \frac{3}{2}\left(i\alpha W_1 + \beta H^{(1)}\right) , \\
T_m &= \alpha\, \partial_m (2A - \phi + \log\alpha) + \alpha \left[W_{4\,m} + \frac{i}{2} J_m{}^n (W_5 - W_4)_n\right] + \frac{1}{2} J_{mn}\beta H^{(3)}_n .
\end{aligned}
\tag{3.24}
$$

For both theories the supersymmetry constraints on the second dilatino and gravitino yield the same expressions with $\alpha \leftrightarrow \beta$, $F_{A1} \to F_{A2}$ and $F_{B1} \to F_{B2}$ for IIA and IIB respectively.

Notice that S and T are just the flux and geometric parts of the superpotential.

4. Conditions on supersymmetric vacua

While the analysis of the implications of the supersymmetry differential equations for pure spinors is done better using a different approach and will be developed in the last section, writing the supersymmetry equations as in (3.22) is better suited for analysis of specific backgrounds as it allows to check how the NS matrices Q balance against RR matrices R representation by representation.

IIA			IIB		
$Q_i : 3$	$R_i : 3$	$S_i : 3$	$Q_i : 3$	$R_i : 3$	$S_i : 3$
$Q_{ij} : 6 \oplus \bar{3}$	-	-	$Q_{ij} : 6 \oplus \bar{3}$	$R_{ij} : 6 \oplus \bar{3}$	-
$Q_{\bar{\imath}j} : 1 \oplus 8$	$R_{\bar{\imath}j} : 1 \oplus 8$	$S : 1$	$Q_{\bar{\imath}j} : 1 \oplus 8$	-	$S : 1$

Table 1: Decomposition of S, Q and R in $SU(3)$ representations.

In IIA, the RR sector consists of a zero- and a six-form with one component each, and a two- and a four-form with 15 components each, making a total of 32 components. Under SU(3) $\in$ SO(6), the zero- and six-form are singlet, while the two- and four- form decompose as $15 \to 1 \oplus 3 \oplus \bar{3} \oplus 8$. In IIB, we of course have again a total of 32 components, however now they are distributed between one- and five-form (in the $3 \oplus \bar{3}$ each) and a three-form ($1 \oplus 3 \oplus 6$ + conjugates). They contribute to the supersymmetry equations through the tensors R's and S's. As for the NS sector it is convenient to switch to a holomorphic basis and analyze the matrices R_i, R_{ij} and $R_{\bar{\imath}j}$, so we need to look only at half of the components on each side. Differently from Q's (see (2.4) and (2.9)) which have the same components in IIA and IIB, some of R's are not generic - there is no 6 appearing in IIA side, and no 8 on IIB side. However, R's and S's together have a total of 16 components in both IIA and IIB. We can collect all the representations in table 1.

The first columns represent the NS sectors and are the same for IIA and IIB ; the mirror symmetry (2.10) exchanges the second and the third lines. The other two columns correspond to the RR sector, and the mirror transformation is

$$\begin{aligned} R_{\bar{\imath}j}(\text{IIA}) &\leftrightarrow -R_{ij}(\text{IIB}) \quad (1 \oplus 8 \leftrightarrow 6 \oplus \bar{3}) \\ R_i &\leftrightarrow -R_{\bar{\imath}} \\ S_i &\leftrightarrow S_i \\ S &\leftrightarrow S\,. \end{aligned} \tag{4.1}$$

Since some representations are missing in R's, $Q_{ij}(\text{IIA})$ and $Q_{\bar{\imath}j}(\text{IIB})$ must be zero by themselves. This is the first hint of the fact that the integrability conditions on one of the pure spinors do not receive RR contributions, as we will show in detail in section 5.

Using eqs. (3.23)–(3.24) we will now give an analysis of solutions to the supersymmetry conditions of IIA and IIB on a manifold of SU(3) structure. (These are necessary conditions; Bianchi identities still have to be imposed to make such backgrounds solutions.) As we will see, one of the main difficulties in this analysis is that according to the relation between the normalizations α and β of the two spinors, the equations we get can be dependent or independent. It proves useful to reduce the freedom of these two quantities by the following argument. For an SU(3) structure, the two-form and three-form should satisfy $J \wedge \Omega = 0$ and $i\Omega \wedge \bar{\Omega} = \frac{2}{3!}J^3$. Both relations are left invariant if one redefines $\Omega \to \Omega e^{i\psi}$, with ψ an arbitrary real function. This shifts $W_5 \to W_5 + id\psi$. In the analysis below, we would always find such a spurious contribution to W_5 to all solutions. This freedom can also be expressed as a rescaling of the spinor $\eta_+ \to e^{i\psi}\eta_+$, or as $\alpha \to \alpha e^{i\psi}, \beta \to \beta e^{i\psi}$. In what follows, we fix it by setting $\text{Arg}(\alpha) + \text{Arg}(\beta) = 0$.

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We start from type IIA theory and we derive the conditions representation by representation.

Scalars: relations among them come from setting $S = 0$, $Q^{(1)}_{\bar{\imath}j} + R^{(1)}_{\bar{\imath}j} = 0$ and $T = 0$. The last condition, $T = 0$, imposes (remember that there is a second set of tensors T, R, S, etc, obtained from (3.23) by exchanging α with β, so $T = 0$ gives two equations): $i\alpha W_1 - \beta H^{(1)} = 0$ and $i\beta W_1 - \alpha H^{(1)} = 0$. From these we see that the only Ansaetze that allow for nonzero scalars in the torsion and H-flux are $\alpha = \pm\beta$. We consider first the case $\alpha \neq \pm\beta$, and we will analyze the case of the equality later. When $\alpha \neq \pm\beta$ we get $W_1 = H^{(1)} = 0$, which means $Q^{(1)}_{\bar{\imath}j} = 0$. The other two conditions, $S = 0$ and $R^{(1)}_{\bar{\imath}j} = 0$, give four (complex) homogeneous equations for the remaining four (real) RR scalars, $F^{(1)}_0$, $F^{(1)}_2$, $F^{(1)}_4$ and $F^{(1)}_6$. These are four independent equations except when $\alpha = \pm\beta$. So for $\alpha \neq \pm\beta$, all scalars are zero.

The case $\alpha = \pm\beta$ works differently. As we showed, the condition $T = 0$ allows for nonzero W_1 and $H^{(1)}$ satisfying $W_1 \mp iH^{(1)} = 0$. On the other hand, adding and subtracting the equations coming from $Q^{(1)}_{\bar{\imath}j} + R^{(1)}_{\bar{\imath}j} = 0$ we can get rid of the RR piece, and we get $W_1 \pm 3iH^{(1)} = 0$. So for this particular Ansatz we also obtain $W_1 = H^{(1)} = 0$. But this Ansatz does allow for RR scalars if they are all equal among them: $F^{(1)}_0 = \pm F^{(1)}_2 = F^{(1)}_4 = \pm F^{(1)}_6$.

$8 \oplus 8$: conditions for these representations come from $Q^{(8)}_{\bar{\imath}j} + R^{(8)}_{\bar{\imath}j} = 0$. $Q^{(8)}_{\bar{\imath}j}$ is proportional to W_2, while $R^{(8)}_{\bar{\imath}j}$ contains the non-primitive (1,1) piece of the RR 2-form and the non-primitive (2,2) piece of the 4-form. These are four real homogeneous equations for four real variables (W_2^+, W_2^- , $F^{(8)}_2$ and $F^{(8)}_4$). The determinant of this system is proportional to $\mathrm{Re}(\alpha\bar{\beta})\mathrm{Re}(\alpha^2 + \beta^2)$. Given the fixing of the total phase of α and β that we did above, the determinant is zero only for α/β purely imaginary. In this case, there is a solution $W_2^+ = e^{\phi}\frac{\mathrm{Im}(\alpha^2)}{|\alpha|^2}F^{(8)}_2$, $W_2^- = e^{\phi}\frac{\mathrm{Re}(\alpha^2)}{|\alpha|^2}F^{(8)}_2$, which is a variation on the holomorphic monopole in [13]. For $\alpha = i\beta$ another independent solution appears, $W_2^+ = F^{(8)}_4$. (Of course the two can be combined.) When we are not in any of these special cases, all the 8 vanish; in particular, W_2 is zero, which together with the condition $W_1 = 0$ obtained in the analysis for the scalars, implies the manifold is complex.

$6 \oplus \bar{6}$: as it can be seen from table 1, IIA solutions should satisfy $Q_{ij} = 0$, which means in particular $Q^{(6)}_{ij} = 0$. This gives again two homogeneous equations ($(\alpha W_3 + i\beta H^{(6)})_{ij} = 0$ and the same with α and β exchanged) that have nontrivial solution only when $\alpha = \pm\beta$. So, for $\alpha \neq \pm\beta$, $W_3 = H^{(6)} = 0$, while for $\alpha = \pm\beta$ we get $W_3 = \pm *_6 H^{(6)}$.

$3 \oplus \bar{3}$: $Q^{(3)}_{ij} = 0$ sets $\alpha W_4 + i\beta H^{(3)} = 0$, and the same with α and β exchanged. So again, for $\alpha \neq \pm\beta$ both W_4 and $H^{(3)}$ are zero, while for $\alpha = \pm\beta$ we get $W_4 = \pm iH^{(3)}$. For the latter, we get that all the RR vectors are zero and $W_5 = 2W_4 = \pm 2iH^{(\bar{3})} = 2\bar{\partial}\phi$, a condition familiar from [10, 11].

The case in which $\alpha \neq \pm\beta$ is more intricate. Some of the many equations can be recombined right away. In particular, one gets that the ratio α/β is a *constant.* This fact is strikingly different from the IIB case we will analyze next; we will comment on this

<table>
<tr><th>IIA</th><th>$\alpha = \pm\beta$</th><th colspan="2">$\alpha = ik\beta$</th></tr>
<tr><td rowspan="2">1</td><td colspan="3">$W_1 = H^{(1)} = 0$</td></tr>
<tr><td>$F_0^{(1)} = \pm F_2^{(1)} =$
$F_4^{(1)} = \pm F_6^{(1)}$</td><td colspan="2">$F_{2n}^{(1)} = 0$</td></tr>
<tr><td rowspan="2">8</td><td rowspan="2">$W_2 = F_2^{(8)} = F_4^{(8)} = 0$</td><td>generic k</td><td>$k = 1$</td></tr>
<tr><td>$W_2^+ = e^\phi \frac{\mathrm{Im}(\alpha^2)}{|\alpha|^2} F_2^{(8)}$
$W_2^- = e^\phi \frac{\mathrm{Re}(\alpha^2)}{|\alpha|^2} F_2^{(8)}$</td><td>$W_2^+ = e^\phi \frac{\mathrm{Im}(\alpha^2)}{|\alpha|^2} F_2^{(8)} + e^\phi F_4^{(8)}$
$W_2^- = e^\phi \frac{\mathrm{Re}(\alpha^2)}{|\alpha|^2} F_2^{(8)}$</td></tr>
<tr><td>6</td><td>$W_3 = \pm *_6 H^{(6)}$</td><td colspan="2">$W_3 = H^{(6)} = 0$</td></tr>
<tr><td>3</td><td>$W_5 = 2W_4 =$
$\pm 2iH^{(\bar 3)} = \bar\partial\phi$
$\bar\partial A = \bar\partial\alpha = 0$</td><td colspan="2">$F_2^{(\bar 3)} = 2iW_5 = -2i\bar\partial A = \frac{2}{3}i\bar\partial\phi,\ W_4 = 0$</td></tr>
</table>

Table 2: Conditions for type IIA susy vacua with $SU(3)$ structure.

difference later. The remaining equations form a system whose determinant is proportional this time to $\mathrm{Re}(\alpha\bar\beta)$. As above, this vanishes for α/β purely imaginary. The solution is in this case $F_2^{(\bar 3)} = \frac{2}{3}i\bar\partial\phi$, $F_4^{(\bar 3)} = 0$, $W_4 = 0 = H^{(3)}$, $W_5 = \frac{1}{3}\bar\partial\phi$, $\bar\partial A = -\frac{1}{3}\bar\partial\phi$. These conditions are those in [13] again.

The table 2 is a summary of what we obtained. In the vector representation (the last row) we have only written the fluxes that are not zero.

From the analysis of the vectors we have that α/β is a constant. Depending on what this constant is, we have different solutions.

We see that essentially the only two supersymmetric solutions of IIA are given by the common sector [10, 11] and by the holomorphic monopole [13], with some variant. There is a natural way of seeing that these two should be the only allowed classes. Let us start from supersymmetric M-theory compactifications. It was shown [14] that these are given by seven-dimensional manifolds with SU(3) structure. This is similar to SU(3) structure in six dimensions, but it also includes a vector v. Reducing to six dimensions will involve another vector e^7, which has to be Killing. In general, e^7 and v together give rise to an SU(2) structure in six dimensions, which is not what we are considering here. If we want an SU(3) structure, there are two possibilities. Either v and e^7 actually coincide, or the SU(3) structure in seven dimensions degenerates to a G_2 structure. The former case gives by reduction [15] the first column of table 2. In the latter case, a G_2 structure does not actually allow for fluxes [14], that is, we are forced to G_2 *holonomy* manifolds. Upon reduction, this leads to the holomorphic monopole geometry [13] of the second column.

The same analysis can be repeated for the type IIB theory.

Scalars: the condition $S = 0$ sets $F_3^{(1)} = 0$, while $T = 0$ together with $Q_{\bar\imath j} = 0$, which annihilate two different combinations of W_1 and $H^{(1)}$, set both of them to zero. The fact that all scalars are zero in supersymmetric IIB solutions has already been noticed in [16, 17].

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$8\oplus 8$: $Q^{(8)}_{\bar{i}j}=0$ sets $W_2=0$. This condition can be easily obtained just by noticing that there is no 8 representation in H, and neither there is in the RR fluxes in IIB. $W_2=0$, together with the condition $W_1=0$, mean that in IIB the complex structure is always integrable. This conclusion, which was previously obtained in [17], is straightforward to get from (3.24).

$6\oplus\bar{6}$: the condition $Q^{(6)}_{ij}+R^{(6)}_{ij}=0$ gives two complex equations for three complex variables W_3, $F_3^{(6)}$ and $H_3^{(6)}$, so we can write two of them as as a function of the third one:

$$\begin{aligned}
(\alpha^2-\beta^2)\,W_3 &= e^{\phi}\,2\alpha\beta\,F_3^{(6)} \\
(\alpha^2+\beta^2)\,W_3 &= -2\alpha\beta\,*_6\,H^{(6)} \\
(\alpha^2-\beta^2)\,H^{(6)} &= e^{\phi}\,(\alpha^2+\beta^2)\,*_6\,F_3^{(6)}\,.
\end{aligned} \tag{4.2}$$

Only for the cases $\alpha=0$ or $\beta=0$; $\alpha=\pm\beta$ and $\alpha=\pm i\beta$ one of the three vanishes (W_3 for the first two cases, while $F_3^{(6)}$ and $H_3^{(6)}$ for the last two respectively). These three cases correspond to well known solutions (called respectively type B, A and C in [18]), as we will discuss below.

$3\oplus\bar{3}$: as opposed to IIA, we cannot isolate two equations which naturally separate the case $\alpha=\pm\beta$ from the others. Neither we find an equation imposing $\alpha/\beta=const.$. One thing that one can do is to look at the three special cases singled out by looking at the 6, and thus imposing by hand $\alpha=\pm\beta$ (case A), $\alpha=0$ or $\beta=0$ (case B), $\alpha=\pm i\beta$ (case C). In all these cases we can analyze the most general solutions for the vectors.

Another thing one can do is to look for solutions which corresponds to generic values of α and β. In this case all the components of the torsion and the fluxes are non-zero and proportional to $\bar\partial\beta$

$$\begin{aligned}
&e^{\phi}F_3^{(\bar 3)}=\frac{-8\alpha}{3\alpha^2+\beta^2}\bar\partial\beta\,, \qquad W_4=\frac{4(\alpha^2+\beta^2)^2}{\beta(\alpha^2-\beta^2)(3\alpha^2+\beta^2)}\bar\partial\beta\,, \quad \bar\partial A=-\frac{2(\alpha^2-\beta^2)}{\beta(3\alpha^2+\beta^2)}\bar\partial\beta\,,\\
&e^{\phi}F_5^{(\bar 3)}=\frac{4i(\alpha^2+\beta^2)}{\beta(3\alpha^2+\beta^2)}\bar\partial\beta\,, \qquad W_5=\frac{2(3\alpha^2+\beta^2)}{\beta(\alpha^2-\beta^2)}\bar\partial\beta\,, \qquad \bar\partial\phi=\frac{16\alpha^2\beta}{(3\alpha^2+\beta^2)(\alpha^2-\beta^2)}\bar\partial\beta\,.\\
&H^{(\bar 3)}=\frac{-8i\alpha(\alpha^2+\beta^2)}{(\alpha^2-\beta^2)(3\alpha^2+\beta^2)}\bar\partial\beta\,,
\end{aligned} \tag{4.3}$$

Moreover the functions α and β are related to the warp factor by $A=\log(|\alpha|^2+|\beta|^2)$, as expected for a supersymmetric IIB compactification [17]. This is an interpolating solution similar to that in [18]. Indeed, although the explicit expression for the fields are different due to the different choices for how to fix the total phase of α and β, a straightforward computation shows that the relations among the fields are actually the same. Notice also that the ratios among the various fields, for example the one between W_4 and $H^{(\bar 3)}$, or between $H^{(\bar 3)}$ and $F_3^{(\bar 3)}$, are singular exactly for the special values of α and β selected above : $\alpha=\pm\beta$, $\alpha=\pm i\beta$ and $\alpha=0$ or $\beta=0$. This seems to suggest that this general solution could interpolate not only between type C and B, as in [18], but also between A and B.

<table>
<tr><th>IIB</th><th>$\alpha = \pm\beta$ (A)</th><th>$\alpha = 0$ or $\beta = 0$ (B)</th><th>$\alpha = \pm i\beta$ (C)</th><th>interp.</th></tr>
<tr><td>1</td><td colspan="4">$W_1 = F_3^{(1)} = H^{(1)} = 0$</td></tr>
<tr><td>8</td><td colspan="4">$W_2 = 0$</td></tr>
<tr><td>6</td><td>$F_3^{(6)} = 0$
$W_3 = \mp * H^{(6)}$</td><td>$W_3 = 0$
$e^{\phi} F_3^{(6)} = *H^{(6)}$</td><td>$H^{(6)} = 0$
$W_3 = \pm e^{\phi} * F_3^{(6)}$</td><td>(4.2)</td></tr>
<tr><td rowspan="2">3</td><td rowspan="2">$\bar{W}_5 = 2W_4$
$= \pm 2iH^{(\bar{3})} = 2\bar{\partial}\phi$
$\bar{\partial}A = \bar{\partial}\alpha = 0$</td><td>$e^{\phi} F_5^{(\bar{3})} = \frac{2}{3} i\bar{W}_5 = iW_4$
$= -2i\bar{\partial}A = -4i\bar{\partial}\log\alpha$
$\bar{\partial}\phi = 0$</td><td rowspan="2">$\pm e^{\phi} F_3^{(\bar{3})} = 2i\bar{W}_5$
$= -2i\bar{\partial}A$
$= -4i\bar{\partial}\log\alpha$
$= -i\bar{\partial}\phi$</td><td rowspan="2">(4.3)</td></tr>
<tr><td>$e^{\phi} F_1^{(\bar{3})} = 2e^{\phi} F_5^{(\bar{3})}$
$= i\bar{W}_5 = iW_4 = i\bar{\partial}\phi$</td></tr>
</table>

Table 3: Conditions for type IIB susy vacua with SU(3) structure.

To conclude the analysis of IIB, we summarize the main features of the three special cases A,B and C. As in type IIA, quantities not mentioned in the table in a given representations are vanishing. The cases shown are this time naturally singled out, but do not exhaust the possibilities. As we have argued above (and as it was found in [18]) there exist solutions that interpolate between the ones shown.

Before moving on, let us identify the columns of table 3. The first represents Strominger's solution [11]. The second column has two sub-cases. The first is a conformal rescaling of a Calabi-Yau metric, with constant dilaton. Klebanov-Strassler solution for the deformed conifold falls into this class [19]–[21]. The second, if one chooses $F_3^{(3)} = H^{(3)} = 0$, corresponds to F-theory on a Calabi-Yau four-fold. Case C is the S-dual of the purely NS solution of case A, a well-known example of which is Maldacena-Nuñez solution [22]. The metric here is the same as for the case A scaled by $\exp(\phi)$, and $\phi \to -\phi$, $H_3 \to -F_3$.

As already mentioned, differently from table 2 for IIA, here we have presented just some special solutions rather than a classification. There is another big difference between the two theories. In IIA, the ratio α/β was constant, and this is not so in IIB. (This ratio is indeed a non constant function in interpolating solutions.) One might wonder how can these differences be compatible with mirror symmetry. The answer is that the freedom present in the spinor Ansatz in IIB has to be reflected somewhere in IIA, but not necessarily in the spinor Ansatz, given that we have not determined here how this maps under mirror symmetry. The mirror of the A case (which is naturally singled out in both theories) corresponds to the same Ansatz on the IIA side. But the mirror of type B class does not neccesarily have the same Ansatz in IIA, and neither does that of type C. Both B and C may be mapped after all to the same Ansatz (the same would then be true for the interpolating solution). A rough argument for this comes comparing with branes on Calabi-Yau manifolds. A D3 brane extended over Minkowski space is in class B, whereas a D5 falls in class C. These two wrap a point and a two-cycle respectively. In a Calabi-Yau, both these branes would be mapped by mirror symmetry to a brane wrapping a special lagrangian three-cycle and therefore the corresponding source is a D6. These are described by the monopole background which is so clearly singled out in second column of table 2.

Following the transformations of the Q's is the practical criterion for determining the mirrors. We observe that other than the $\alpha = \pm\beta$ solution for IIB, all the others have $Q_{ij} \neq 0$. Thus, they require a mirror with non-integrable complex structure. The fact that on type IIA side there is a unique solution leaves us no choice but to conclude that the monopole geometry is the universal mirror for all IIB solutions. Given that for the former $dJ = 0$, and thus the geometry is symplectic, it is a rather natural mirror to the IIB geometry, which is always complex.

5. Pure spinor equations

In this section we will finally derive the pure spinor equations promised in the introduction, and thereby justify the title of the paper. First of all we will introduce the formalism of Clifford(6, 6) spinors, to justify the use of the expression e^{iJ} often mentioned above (see for example (2.13)). Then we will give a brief mathematical introduction to the use of pure spinors in the context of generalized complex geometry. This will motivate us to look for differential equations on the pure spinors defined on a SU(3) structure manifold, which we finally present in equations (5.8)–(5.11).

5.1 Pure spinors

We start by introducing the formalism of pure spinors. We will also comment on how to obtain torsions from them.

Clifford(6, 6) is an algebra of matrices λ^m, ρ_n that obey

$$\{\lambda^m, \lambda^n\} = 0\,, \qquad \{\lambda^m, \rho_n\} = \delta^m_n\,, \qquad \{\rho_m, \rho_n\} = 0\,.$$

We have chosen two different symbols, λ and ρ, instead of the more commonly used γ^m and γ_m, to emphasize that these matrices are independent, they cannot be obtained from each other by raising and lowering indices with the metric. So the number of gamma matrices is twice the dimension of the manifold, in our case twelve. The representation of this algebra which is usually taken, and to which we will stick, is on the vector space of formal sums of forms of all degrees, $\oplus_{i=1}^{6}\Lambda^i T^*$. Then $\lambda^m = dx^m\wedge$, and $\rho_n = \iota_{\partial_n}$ (see footnote 7 for the explicit action of ι_{∂_n}).

A pure spinor is one whose annihilator is a six-dimensional space in the twelve-dimensional algebra Clifford(6, 6). On an SU(3) structure manifold there are two natural pure spinors. One is simply Ω, which is annihilated by λ^i ($\Omega_{jkl}\, dz^i \wedge dz^j \wedge dz^k \wedge dz^l = 0$) and $\rho_{\bar\imath}$ ($\Omega_{\bar\imath kl} = 0$). Another, which might seem more exotic, is $e^{iJ} \equiv 1 + iJ - 1/2 J\wedge J - i/6 J\wedge J\wedge J$. It is annihilated by $\rho_m + iJ_{mn}\lambda^n$, as it is easy to check using $J_m{}^n J_n{}^p = -\delta_m{}^p$.

We will also use the familiar fact that we can map a form (or a formal sum of them) to an element of the *usual* Clifford algebra, Clifford(6):

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad \longleftrightarrow \qquad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1\ldots i_k} \gamma^{i_1\ldots i_k}_{\alpha\beta}\,. \tag{5.1}$$

An object in Clifford(6) can also be seen as a bispinor, since it has two free spinor indices. So we have realized Clifford(6, 6) spinors as bispinors, which are more useful in string theory. We will see that it is crucial that $e^{i\not{J}}$ and $\not\Omega$ can be reexpressed in terms of tensor

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products of η, as stated in the introduction. Another useful technical fact is that one can realize λ and ρ also as combinations of the more familiar γ's acting on the left and on the right of a bispinor. For example, $\lambda^m C^{(k)} \longleftrightarrow \frac{1}{2}(\gamma^m \not{C}^{(k)} \pm \not{C}^{(k)}\gamma^m)$ when the plus (minus) sign corresponds to k even (odd).

All this technical machinery is mainly needed to give a meaning to the expression e^{iJ}. We have just seen that it is a pure spinor, as it is Ω. Thus on a manifold of SU(3) structure there are always two pure spinors. In this formulation, it is not unnatural to think there might be a mirror symmetry exchanging these two:

$$e^{iJ} \longleftrightarrow \Omega\,.$$

This was indeed the formulation of mirror symmetry in [4]. In the first part of this paper, we have lent credence to this conjecture by showing that supergravity (its supersymmetry transformations) can be rewritten in terms of these two pure spinors alone (or rather under their bispinor counterparts, eq. (5.1)). We have also given explicitly the exchange under which the two are symmetric, eq. (4.1). Later in this section we will also show how the supergravity equations imply differential equations in which they appear symmetrically. For now, let us comment a moment about what is the interpretation of the R_{mn} from the point of view of pure spinors.

In the appendix we use Fierz identities crucially, see for example equation (A.8). In that conventional treatment, one uses a basis $\gamma^{m_i\ldots m_k}$ for expanding an arbitrary element of the Clifford(6) algebra. These are obtained from a trivial vacuum 1 acting with all the possible gamma matrices. However, this procedure can be repeated just as well replacing 1 with another pure spinor. In mathematical terms, Gualtieri [7, section 3.6] introduces a filtration of Clifford algebra by the number of gamma matrices acting on a given pure spinor. In a more concrete language, this yields another basis for the Clifford algebra. For example, with Ω as a Clifford vacuum, the basis is given by $\gamma_{i_1\ldots i_k}\Omega\gamma_{j_1\ldots j_l}$: all the possible holomorphic gammas from each side of Ω. More explicitly, the first few read Ω, $\gamma_i\Omega$, $\Omega\gamma_i$, $\gamma_i\Omega\gamma_i$. Analogously, taking e^{iJ} as a Clifford vacuum, the basis is $\gamma_{\bar{i}_1\ldots\bar{i}_k}e^{iJ}\gamma_{j_1\ldots j_k}$. One can use these bases equally well to derive Fierz identities. For the usual basis $\gamma^{m_1\ldots m_k}$, the coefficients of the expansion of a bispinor F are $\mathrm{Tr}(C\gamma_{m_1\ldots m_k})$. (If instead of F we have $\eta_+\otimes\eta_\pm^\dagger$, then we find (A.8)). If we now use one of our new bases, say the one relative to Ω, the first coefficients of the expansion would now look

$$\mathrm{Tr}(F\bar{\Omega})\,,\qquad \mathrm{Tr}(F\gamma^i\bar{\Omega})\,,\qquad \mathrm{Tr}(F\bar{\Omega}\gamma^i)\,,\qquad \mathrm{Tr}(F\gamma^i\bar{\Omega}\gamma^j)\,.$$

These, along with their analogues built using e^{iJ}, are nothing but the S, S_m, R_m, R_{mn} introduced in the previous sections, as detailed in the appendix. This gives us an intuitive explanation of the exchange (4.1). Since the pure spinors are exchanged, $e^{iJ}\leftrightarrow\Omega$, all the tower of states built by Clifford action from them will be exchanged (the "filtration"); the coefficients of the expansions will be exchanged too, and this is what (4.1) provides.

Let us finally also comment on how to get torsions from pure spinors. Schematically one has

$$D_m e^{iJ} = \mathrm{Im}\left(q_m^{(2)}\cdot\Omega\right)\,,\qquad D_m\Omega = -iq_m\Omega - \mathrm{Im}\left(q_m^{(2)}\cdot e^{iJ}\right)\,. \tag{5.2}$$

In these equations, $q_m^{(2)}\cdot$ is the usual Clifford product of q_{mn} using only second index.

5.2 Pure spinors in generalized complex geometry

We will now see how pure spinors are used in the context of generalized complex geometry. The basic thing we want to show is that pure spinors can be used instead of *generalized almost complex structures.*

The idea behind generalized complex geometry is to consider the direct sum of the tangent and cotangent bundle, rather than the tangent bundle itself, and to generalize the standard machinery of complex geometry.

If ordinary almost complex structures J are bundle maps from T to itself that square to -1, *generalized* almost complex structures $\mathcal{J}$ are maps of $T \oplus T^*$ to itself that square[9] to -1

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix}, \tag{5.3}$$

where $J : T\mathcal{M} \to T\mathcal{M}$, $P : T^*\mathcal{M} \to T\mathcal{M}$, $L : T\mathcal{M} \to T^*\mathcal{M}$ and $K : T^*\mathcal{M} \to T^*\mathcal{M}$.

From this expression it is easy to see that usual complex structures are naturally embedded into $\mathcal{J}$: they correspond to the choice

$$\mathcal{J}_1 \equiv \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}, \tag{5.4}$$

with J_m^n an almost complex structure. Another example of generalized almost complex structure can be built Using a non degenerate two-form ω,

$$\mathcal{J}_2 \equiv \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{5.5}$$

There is a natural integrability condition for generalized almost complex structures, analogous to the integrability condition for usual almost complex structures. For the usual complex structures integrability, namely the vanishing of the Nijenhuis tensor (or in our terms, $q_{i\bar{\jmath}} = 0$), can be written as a condition on the Lie bracket on T. For generalized almost complex structures the Lie bracket is replaced by a certain bracket on $T \oplus T^*$, called the *Courant* bracket, which does not satisfy Jacobi in general, but which does on the i-eigenspaces of a $\mathcal{J}$.

In case these new conditions are fulfilled, we can drop the "almost" and speak of generalized complex structures. It is interesting to see what this condition is for the two examples above. For the one we called $\mathcal{J}_1$, which was built from an almost complex structure, integrability coincides with the ordinary meaning, thus making it a complex structure. For $\mathcal{J}_2$, which was built from a two-form ω, the condition becomes $d\omega = 0$, thus making ω into a *symplectic* form.

These two examples are not exhaustive, and the most general generalized complex structure *interpolates* between complex and symplectic manifolds. It is immediately clear that this formalism then must be useful for mirror symmetry: although for the physical string mirror symmetry is an exchange of Calabi-Yaus, for the topological string it can be formulated as sending symplectic manifolds into complex ones, and vice versa.

[9] As for an almost complex structure, $\mathcal{J}$ must also satisfy the hermiticity condition $\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}$, with the respect to the natural metric on $T \oplus T^*$, $\mathcal{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Now, what is more immediately relevant in our context is that generalized complex geometry can be reformulated in terms of pure spinors. First of all there is an algebraic correspondence between a generalized almost complex structure $\mathcal{J}$ and a pure spinor φ. As an example, $\mathcal{J}_1$ above is sent by this correspondence to a section of the bundle of $(3,0)$ forms, which in the case of SU(3) structures exists: we have called it Ω so far. The other example, $\mathcal{J}_2$ is sent to e^{iJ}, where we have renamed the symplectic form ω as J, as in the rest of the paper.

Under this correspondence, the integrability condition for a generalized almost complex structure is equivalent to the condition that the spinor φ is closed. This means that every degree of φ is separately closed (remember that a Clifford$(6,6)$ spinor is a formal sum of forms).[10] Manifolds on which a closed pure spinor exists were called generalized Calabi-Yaus by Hitchin [6].

There is also a possibility of adding a three-form H to the story. Using a three-form, the Courant bracket can be modified, and with it the integrability condition. Not surprisingly, this also corresponds to a modification of the condition on the pure spinor, which now becomes

$$(d + H\wedge)\varphi = (v\llcorner + \xi)\varphi \tag{5.6}$$

for some v and ξ (compare with footnote 10). If we decompose φ in forms, $\sum \varphi_{(k)}$, the condition means this time that $d\varphi_{(k)} + H \wedge \varphi_{(k-2)} = v\llcorner\varphi_{(k+2)} + \xi \wedge \varphi_{(k)}$.

5.3 Supersymmetry equations for pure spinors

In this section we will finally use the work done on supergravity to derive equations on the two pure spinors. These equations do not encode all the information coming from the supersymmetry conditions. They are rather the counterpart of the internal gravitino, in that they encode derivatives of objects that can be used to define the structure. They capture the information about the intrinsic torsion of the manifold; but in general as we have seen there are more conditions, equaling components of fluxes (and derivatives of the dilaton and warping) among each other. A natural question for a string theorist is whether (5.6) is relevant in any way to compactifications with fluxes. The first natural example to look at is of course the case with only H present [10, 11] In section 5.2 we have already noticed that the manifold is complex; with little more effort, one gets that $e^{2\phi}\Omega$ is closed. So in that case the manifold is generalized complex.

In this, however, H did not enter. It is natural to guess that the condition on e^{iJ} will be involving H, and maybe even that it would be as in (5.6).

Unfortunately, this guess turns out to be incorrect. One way to see this is to use the so-called torsional equation which can be seen to follow from the equations on the intrinsic torsions (for example). This reads $\partial J = -iH^{(2,1)}$. Then we have $de^{iJ} = \left(H^{(2,1)} - H^{(1,2)}\right) e^{iJ}$, falling short of proving (5.6) for e^{iJ}.

[10] More precisely, the condition is that there exist a vector v and a one-form ξ such that $d\varphi = (v\llcorner + \xi)\varphi$.

We may, then, propose another point of view. Hitting (2.13) with an extra λ (a wedging), one gets an equation

$$\left(d + \frac{1}{4} H_{mnp} dx^m dx^n \iota^p\right) \varphi = 0 \,. \tag{5.7}$$

So, H acts by contraction of one index and wedging of the other two. Notice also that this operator raises the degree of the forms in φ by one, the same as d. This form of the equations has then the advantage that H acts in the same way as d, which is not the case in (5.6). But it would seem at this point that we have many options, and it is not clear how to pick the most relevant or interesting.

Introduction of RR fluxes makes the story very different. A priori, we might again consider many combinations and get many different ways for H to act. However, we have found only one choice for which the RR flux only contributes to one of the two spinor equations. Here is a schematical description of how this computation is performed, for the pure spinor e^{-iJ}.

Our strategy is to use the Clifford map (5.1) between pure spinors and bispinors. As we mentioned in the introduction, the pure spinors have a particularly simple form under this map. For example, $e^{-iJ} = 8\eta_+ \otimes \eta_+^\dagger$, see eq.(A.9). The exterior derivative $d(e^{-iJ})$ can be re-expressed in the bispinor picture as the anticommutator

$$\{\gamma^m, D_m(\eta_+ \otimes \eta_+^\dagger)\} \,.$$

The covariant derivative here is meant to be a bispinor covariant derivative, which corresponds to the ordinary covariant derivative of forms under the Clifford map, and which anyway reduces to exterior derivative when we fully antisymmetrize, as usual. To compute this object, one can use Leibniz rule for the covariant derivative of the bispinor, reducing it to $\{\gamma^m, D_m(\eta_+) \otimes \eta_+\}$ plus its complex conjugate. The covariant derivative of the spinor con now be read off (A.3) in IIA or (A.6) in IIB. Actually, when γ^m acts on the left, one reconstructs the Dirac operator, which is better to read directly from (A.3) or (A.6) in IIA and IIB, respectively. Substituting gives

$$\begin{aligned}
\text{IIA}: \quad & -\left[\not\partial(2A - \phi + \log\alpha) + \frac{\beta}{4\alpha}\not{H}\right] \eta_+ \otimes \eta_+^\dagger - \left(\partial_m \alpha + \frac{\beta}{4\alpha} H_m\right) \eta_+ \otimes \eta_+^\dagger \gamma^m \,, \\
\text{IIB}: \quad & -\left[\not\partial(2A - \phi + \log\alpha) - \frac{\beta}{4\alpha}\not{H}\right] \eta_+ \otimes \eta_+^\dagger - \\
& - \left(\partial_m \alpha - \frac{\beta}{4\alpha} H_m + -\frac{i}{4\alpha} e^\phi \not{F}_{B\,1}\gamma_m\right) \eta_+ \otimes \eta_+^\dagger \gamma^m \,,
\end{aligned}$$

plus again the complex conjugates. Notice that in IIA F has disappeared. This is because it would have multiplied $\gamma_m \eta_- \otimes \eta_+^\dagger \gamma^m$. This expression is zero because $\eta_- \otimes \eta_+^\dagger = -\frac{i}{8}\bar{\not\Omega}$, (see again (A.9)) and $\gamma_m \gamma^{npq} \gamma^m = 0$ in six dimensions. This technical circumstance is what allows us to make F disappear in one of the pure spinor equations for both IIA and IIB. As for the other expression, $\gamma_m \eta_+ \otimes \eta_+^\dagger \gamma^m$, it can be re-expressed in terms of $\eta_- \otimes \eta_-^\dagger$, but it is not vanishing.

It is now only required to go from the bispinor picture back to the form picture, inverting the Clifford map (5.1). Analogous computations can be performed for $d\Omega$. The final results are as follows. In type IIA we have

$$
\begin{aligned}
e^{-f} d\left(e^{f} e^{iJ}\right) &= -\frac{1}{2}\frac{\mathrm{Re}(\alpha\bar\beta)}{|\alpha|^2+|\beta|^2} H \bullet e^{iJ}\,, && (5.8)\\
e^{-g} d\left(e^{g}\Omega\right) &= -\frac{1}{4}\frac{\beta^2+\alpha^2}{2\alpha\beta} H\bullet\Omega \,- && (5.9)\\
&\quad -\frac{e^{\phi}}{16}\frac{1}{2\alpha\beta}\left(F\cdot\left(-\frac{1}{4}e^{-iJ}+1+i\mathrm{vol}\right)-\left(-\frac{1}{4}e^{iJ}+1-i\mathrm{vol}\right)\cdot F^*\right),
\end{aligned}
$$

and in type IIB

$$
\begin{aligned}
e^{-f} d\left(e^{f} e^{iJ}\right) &= \frac{1}{2}\frac{\mathrm{Re}(\alpha\bar\beta)}{|\alpha|^2+|\beta|^2} H \bullet e^{iJ} \,- && (5.10)\\
&\quad -i\frac{e^{\phi}}{16}\frac{1}{|\alpha|^2+|\beta|^2}\left(F\cdot\left(-\frac{1}{4}e^{-iJ}+1+i\mathrm{vol}\right)-\left(-2e^{-iJ}+1+i\mathrm{vol}\right)\cdot F\right),\\
e^{-g} d\left(e^{g}\Omega\right) &= \frac{1}{4}\frac{\beta^2+\alpha^2}{2\alpha\beta} H\bullet\Omega\,. && (5.11)
\end{aligned}
$$

In both cases $f = 2A - \phi + \log(|\alpha|^2 + |\beta|^2)$ and $g = 2A - \phi + \log(\alpha\beta)$, and $F \equiv (|\alpha|^2 - |\beta|^2)F_+ + (\alpha\bar\beta - \bar\alpha\beta)F_-$, where F_+ is the hermitean piece of the RR total form ($F_+ = F_0 + F_4$ in IIA, $F_+ = F_1 + F_5$ in IIB) and F_- is the anti-hermitean piece ($F_- = F_2 + F_6$ in IIA and $F_- = F_5$ in IIB). Since we wrote these equations with forms rather than bispinors, we explicitly denoted the Clifford product between forms by a $\cdot$; vol is the volume form, whose image under the Clifford map would be $i\gamma$. The operator $H\bullet$ is the same for all equations and is defined by

$$
H\bullet \equiv H_{mnp}\left(dx^m dx^n \iota^p - \frac{1}{3}\iota^m\iota^n\iota^p\right). \tag{5.12}
$$

Although the RR piece is not very nice, it has a similar form in both cases too. Most importantly, the action of the NS sector is always the same. If the symmetry between the two theories was more or less guaranteed by the analysis of the previous sections, nothing a priori guaranteed that the operator $H\bullet$ would be the same also for both pure spinors, if not a vague analogy with equation (5.7). Notice that in that case we had been driven to a different choice of operator. But it was not the only one we could have put in an (5.7), as we see now from specializing equations (5.8)–(5.11) to the case with no RR.

Given the mathematical discussion, it is natural to wonder if the operator $H\bullet$ we found has a realization in terms of a twisting of the Courant bracket. Note however that the combination $d + H\bullet$ does not square to zero this time.

With this caveat (or technical clarification) in mind, we will call any action of H-flux a twisting. Then it is easy to tell what the main outcome of the equations (5.8)–(5.9) for IIA and (5.10)–(5.11) for IIB is. In each case, we have one pure spinor equation that contains an exterior derivative and H-twist. Thus, having a twisted closed pure spinor, or in other words twisted generalized Calabi-Yau, is a necessary condition for having an $\mathcal{N} = 1$ vacuum. All the backgrounds constructed so far satisfy this condition.

Strictly speaking we have proven this statement only for manifolds with SU(3) structure since the existence of this structure was assumed in writing the covariant derivatives on the supersymmetry parameter. We would like to argue though that this simplifying technical assumption can be dropped and the result may hold in more general cases. Going beyond the SU(3) structure is not a self-purpose - for example both for IIA and for IIB there are known vacua which correspond to compactifications on non-spin (yet indeed generalized Calabi-Yau) manifolds and the structure group for these will not fit into SU(3). To see the validity of the formulae for the U(3) structure case, we can go back to (2.11) and the footnote 7: reformulating our results in terms of the covariant derivatives of the fundamental form J is the best way to see that at least conceptually one can do without Ω or a well-defined η. Note that treating W_5 not as a well-defined one-form but as a connection would still allow to write (2.3) making use of the spin^c structure of the manifold. We leave a more thorough discussion of global (and non-geometric) aspects for future work.

A comment is due about the equivalence of (3.22) with the pure spinor equations. Clearly most of the information is contained in the part related to the internal gravitino and as already discussed this is captured by (5.8)–(5.11). The two vector-like equations simply serve to define the warp factor and the dilaton in terms of the geometry and flux data and then have to be added to the two pure spinor equations. As already noticed the remaining conditions $S = 0 = T$ comes from the geometric and flux contributions to the superpotential. We conclude this section by noticing that collecting all pieces, the superpotential may indeed be written using pure spinors (for the derivation see [23]):

$$W = S + T = e^{\phi} e^{-B} \varphi_1 d(e^B (e^{-\phi} \varphi_2 + iC)) \,.$$

The RR gauge field C here stands for the formal sum of all forms and its rank being odd or even is correlated with the rank of φ_2. In other words, $\varphi_1 = \exp(iJ)$ and $\varphi_2 = \Omega$ for IIA and the other way around for IIB. It is not hard to verify explicitly that this expression contains all the known contributions to the superpotential and nothing else.

5.4 Topological strings and auxiliary fields.

It is now natural to wonder, what the physical meaning of the decoupling of one of the two pure spinors from the RR fields is.

A natural answer can be found in the context of topological strings. It is well-known that the A and B model see only the symplectic and the complex structure of a Calabi-Yau, respectively. We have seen in section 5.2, in the context of generalized complex geometry, that the pure spinor e^{iJ} corresponds to the symplectic structure, and the pure spinor Ω corresponds to the complex structure. So, in our language topological strings only see one of the two spinors.

If we have a nonlinear sigma model with a manifold of SU(3) structure M as target, the requirement of extended supersymmetry will impose certain differential conditions on the structure. Analogously, trying to define A and B models on M will lead to certain differential conditions on the pure spinor e^{iJ} or Ω respectively, via the requirement of

BRST closure. (With non-zero H, this has not been done explicitly so far. The proper way of doing this is most probably using again generalized complex geometry and a framework similar to [24].)

Let us now try to switch RR fields on. It was argued in [1] that their introduction does not modify topological string amplitudes. (The argument is also interesting in the present context, and we will come back to it shortly.) If the topological model does not feel the RR fields, the differential conditions we are finding should not change. This is indeed what we have in equations (5.8) and (5.11).

Let us now come back to the argument that guarantees that topological strings are not affected by RR fields. It goes roughly as follows. Suppose for a moment that we are on a Calabi-Yau, but with RR fields switched on. The superpotential in this case is the usual Gukov-Vafa-Witten one [25, 26]. It was pointed out in [1] that fluxes can be introduced by simply giving vacuum expectation value to the auxiliary fields of the vector superfield. From this, one can also see, by integration over one linear combination of the two θ of $\mathcal{N}=2$ superspace,[11] that the Gukov-Vafa-Witten superpotential is reproduced *without changing the prepotential* $\mathcal{F}_0$. Since the latter is a topological amplitude, this means topological amplitudes are not changed by RR fields. This argument was made more precise in [27] for IIB theory.

What is of more interest here, is that intrinsic torsions can also be realized (analogously to RR fluxes) by giving expectation values to auxiliary fields. This allows to reproduce the extension of the Gukov-Vafa-Witten superpotential including intrinsic torsions that appeared in [3]. More precisely, the auxiliary fields for the vectors in type IIA were argued in [1] to contain torsions in the representations 8 and 1, namely W_1 and W_2. On the other side, the auxiliary fields for the vectors in type IIB were shown in [27] to contain torsions in representations 6 and 3, namely W_3 and W_4. The effective four-dimensional theories should be equal for two mirror compactifications. Thus we are lead to say that mirror symmetry should exchange torsions in 6+3 with torsions in 8+1. This is indeed consistent with the recipe given in [4] and reviewed here.

Moreover, in [27] a three-form superfield is introduced, which has as auxiliary fields W_3 and W_4 (as well as fluxes). Its lowest component is simply Ω and it is given by

$$\Omega(\theta,\tilde{\theta}) = \Omega + \theta^2(dJ+H) + \tilde{\theta}^2(dJ-H) + \theta\tilde{\theta}(F - C_0 H) + \cdots .$$

The logic of the present paper tells that a similar object should exist for IIA except that this time it is an e^{iJ} superfield - namely an object whose lowest component is given by a formal sum of terms. Note that such a structure is also consistent with the flux superpotential for IIA. Moreover the general intuition from mirror symmetry would tell that in $\theta\tilde{\theta}$-term it is more natural to expect an NS contribution from the metric rather than the H-flux. Indeed, following the logic of [27] and using the IIA vacua from section 4, it is possible to construct such an object.

We would like to take the conjecture one step further and motivated by the fact that the superpotential can be written in a unified fashion for IIA and IIB using pure spinors,

[11] This choice will correspond to the α and β in our Ansatz above.

introduce the "pure spinor superfield", which for IIA will have e^{iJ} as its lowest component and for IIB reduces to the three-form superfield of [27]:

$$\varphi_1(\theta,\tilde{\theta}) = \varphi_1 + \theta^2(d+H\bullet)\varphi_2 + \tilde{\theta}^2(d-H\bullet)\varphi_2 + \theta\tilde{\theta}(e^{\phi}d(e^{-\phi}\varphi_2 + iC)\varphi_1) + \cdots \,,$$

where as usually for IIA $\varphi_1 = \exp(iJ)$ and $\varphi_2 = \Omega$ and the inverse for IIB, and C is the total RR field. The $\theta\tilde{\theta}$ component can be obviously changed by linear redefinitions such as $\theta \to \theta + \tilde{\theta}$ to get an alternative form $(H\bullet + C)\varphi_2$, which agrees with the three-form superfield above.

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A. Derivation of the supersymmetry equations in terms of Clifford products

In this section we show how to derive expressions (3.23) and (3.24) for the tensors S, S_m, R_m and R_{mn} introduced in (3.22). We will perform the derivation for IIA; the case of IIB works analogously.

To treat the supersymmetry constraints for IIA and IIB in the most symmetrical way, it is convenient to consider linear combinations of the equations (3.1), (3.4) for the two spinors ϵ_1 and ϵ_2. Inserting the Ansaetze (3.5), (3.9), (3.12) and (3.11) for the metric, the spinors and the RR field strengths, the equation for the space-time components of the gravitino, $\delta\psi_\mu$, can be rewritten as

$$\begin{aligned} \alpha\slashed{\partial} A\eta_+ + \frac{i}{4}e^{\phi}\slashed{F}_{A1}\eta_- &= 0\,, \\ \beta\slashed{\partial} A\eta_+ + \frac{i}{4}e^{\phi}\slashed{F}_{A2}\eta_- &= 0\,, \end{aligned} \tag{A.1}$$

where the coefficients α and β are related to those in (3.12) by

$$\alpha \equiv a + ib \qquad \beta \equiv a - ib\,, \tag{A.2}$$

and F_{A1} and F_{A2} are those defined in (3.20).

Repeating the procedure for the internal gravitino, $\delta\psi_m = 0$, we obtain

$$\alpha D_m(\eta_+) + \left(\partial_m\alpha + \frac{1}{4}\beta\slashed{H}_m\right)\eta_+ + \frac{i}{8}e^{\phi}\slashed{F}_{A1}\gamma_m\eta_- = 0\,, \tag{A.3}$$

and the same expression with $\alpha \leftrightarrow \beta$ and $F_{A1} \leftrightarrow F_{A2}$.

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Similarly, for the modified dilatino equation one has

$$\alpha \not{D}(\eta_+) + \left[\alpha \not{\partial}(2A - \phi + log\alpha) + \frac{1}{4}\beta \not{H}\right] \eta_+ = 0\,, \tag{A.4}$$

and again the same thing with $\alpha \leftrightarrow \beta$.

For completeness we also list the corresponding equations for IIB:

$$\begin{aligned}
\left[\alpha \not{\partial} A + \frac{i}{4} e^{\phi} \not{F}_{B1}\right] \eta_+ &= 0\,, & \text{(A.5)}\\
\alpha D_m(\eta_+) + \left[\partial_m \alpha - \frac{1}{4}\beta H_m - \frac{i}{8} e^{\phi} \not{F}_{B1}\gamma_m\right] \eta_+ &= 0\,, & \text{(A.6)}\\
\alpha \not{D}(\eta_+) + \left[\alpha \not{\partial}(2A - \phi + log\alpha) - \frac{1}{4}\beta \not{H}\right] \eta_+ &= 0\,, & \text{(A.7)}
\end{aligned}$$

where α and β are defined as for IIA, F_{B1} is defined in (3.20), and we have again a second set of equations with $\alpha \leftrightarrow \beta$ and $F_{B1} \leftrightarrow F_{B2}$.

We want to write these equations in terms of Clifford products of the NS/RR fluxes with the two pure spinors e^{iJ} and Ω. As mentioned in the introduction, using Fierz rearrangement

$$\eta_\pm \otimes \eta_+^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_+^\dagger \gamma_{i_1 \dots i_k} \eta_\pm \gamma^{i_k \dots i_1} \tag{A.8}$$

it is possible to express the pure spinors as tensor products of the standard spinor defining the SU(3) structure

$$\begin{aligned}
\eta_\pm \otimes \eta_\pm^\dagger &= \frac{1}{8} e^{\mp i \not{J}}\,,\\
\eta_+ \otimes \eta_-^\dagger &= -\frac{i}{8} \not{\Omega}\,,\\
\eta_- \otimes \eta_+^\dagger &= -\frac{i}{8} \overline{\not{\Omega}}\,.
\end{aligned} \tag{A.9}$$

We can then rewrite equations (A.1), (A.3) and (A.4) in terms of Clifford products by multiplying them to the left by $\eta_\pm^\dagger$ and $\eta_\pm^\dagger \gamma_n$. Take for example the spacetime gravitino variation (A.1), which we want to rewrite in the form (3.22), i.e.

$$\alpha \not{\partial} A \eta_+ - \frac{i}{4} e^{\phi} \not{F}_{A1} \eta_- = S_{A1} \eta_- + (S_{A1\,m} + A_m)\gamma^m \eta_+\,. \tag{A.10}$$

To obtain S_{A1}, we multiply this equality by $\eta_-^\dagger$ to the left on both sides. On the right hand side only $\frac{1}{2} S_{A1}$ remains, while on the left hand side the first term goes away, and the other can be written as

$$\eta_-^\dagger \not{F}_{A1} \eta_- = Tr(\eta_-^\dagger \not{F}_{A1} \eta_-) = \frac{1}{8} Tr(\not{F}_{A1} e^{i\not{J}}) = \frac{1}{2} (\not{F}_{A1} e^{i\not{J}})_0\,. \tag{A.11}$$

In the first equality we inserted a trace since the lhs is a number, while in the second one we have used the cyclic property of the trace and (A.9). Finally in the last step we used the fact that antisymmetric products of gamma matrices are traceless, and that the gammas in six dimensions are 4 x 4 matrices, so Tr$\mathbf{1} = 4$. By $(\dots)_0$ we mean the term in the Clifford product that does not contain any gamma matrix.

Similarly, $S_{A1\,m}$ and A_m are obtained multiplying (A.10) by $\eta_+^\dagger \gamma_p$. On the right hand side we have $\overline{P}_p{}^m S_m$, where $\overline{P}$ is the projector onto antiholomorphic indices $\overline{P}_n{}^m = \frac{1}{2}(\delta_n{}^m + iJ_n{}^m)$, while on the left hand side we get the warp factor term and a contribution from the RR fluxes. To evaluate this contribution, we need the identity

$$\eta_+^\dagger \gamma_p \not{F}_A \eta_- = \mathrm{Tr}(\eta_+^\dagger \gamma_p \not{F}_A \eta_-) = -\frac{i}{8}\mathrm{Tr}(\overline{\not{\Omega}}\gamma_p \not{F}_A) = -\frac{i}{2}(\not{F}_A \overline{\not{\Omega}})_p \,, \tag{A.12}$$

where $(\ldots)_p$ means the term that multiplies γ^p in the Clifford product.

Collecting everything together, the spacetime gravitino variations (A.1) can be written as

$$\frac{i}{4}e^\phi(\not{F}_{A1}e^{i\not{J}})_0\eta_- + \left[\frac{1}{8}e^\phi \mathrm{Re}(\not{F}_{A1}\overline{\not{\Omega}})_m + \alpha\partial_m A\right]\gamma^m\eta_+ = 0\,, \tag{A.13}$$

plus the same equation with $F_{A1} \to F_{A2}$ and $\alpha \to \beta$. From this equation we can immediately read S, S_m and A_m as given in 3.23.

As for the internal gravitino equations (A.3), we want to write them in the form (3.22)

$$Q_m\eta_+ + Q_{mp}\gamma^p\eta_- + R_m\eta_+ + R_{mp}\gamma^p\eta_- \,, \tag{A.14}$$

where Q_m and Q_{mp} summarize the torsions and NS-flux contribution. Their expressions can be found in (3.23).

To get R_m and R_{mp} we should multiply (A.3) by $\eta_+^\dagger$ and $\eta_-^\dagger\gamma_n$ on the left, respectively. Doing this on (A.14), we obtain $\frac{1}{2}R_m$ and $P_n{}^p R_{mp}$, where P is the projector onto holomorphic indices $P_n{}^m = \frac{1}{2}(\delta_n{}^m - iJ_n{}^m)$.

We repeat this procedure for all supersymmetry equations which can now be written in a nice and compact way

$$\frac{i}{4}e^\phi(\not{F}_{A1}e^{i\not{J}})_0\,\eta_- + \left[\frac{1}{8}e^\phi(\not{F}_{A1}\overline{\not{\Omega}})_m + \alpha\partial_m A\right]\gamma^m\eta_+ = 0\,, \tag{A.15}$$

$$\begin{aligned}\frac{i}{8}e^\phi\left[\frac{1}{2}(\not{F}_{A1}e^{i\not{J}})_0 g_{mp} + (\not{F}_{A1}e^{i\not{J}})_{mp} - (\not{F}_{A1\,m}e^{i\not{J}})_p\right]\gamma^p\eta_- +&\\ +iQ_{1\,m}\eta_+ + iQ_{1\,mp}\gamma^p\eta_- + \frac{1}{8}e^\phi(\overline{\not{\Omega}}\not{F}_{A1})_m\eta_+ &= 0\,,\end{aligned} \tag{A.16}$$

$$\begin{aligned}\left[i\alpha q_m + \frac{i}{2}\alpha q_{nr}\Omega^{nr}{}_m + \frac{1}{48}\beta(\not{H}e^{-i\not{J}})_m + \alpha\partial_m(2A - \phi + \ln\alpha)\right]\gamma^m\eta_+ +&\\ +\left[2i\alpha P^{mn}q_{mn} - \frac{i}{24}\beta(\not{H}\Omega)_0\right]\eta_- &= 0\end{aligned} \tag{A.17}$$

for IIA, and for IIB

$$\frac{1}{4}e^\phi(\not{F}_{B1}\not{\Omega})_0\,\eta_- + \left[\frac{i}{8}e^\phi(\not{F}_{B1}e^{-i\not{J}})_m + \alpha\partial_m A\right]\gamma^m\eta_+ = 0\,, \tag{A.18}$$

$$\begin{aligned}iQ_{1\,m}\,\eta_+ + iQ_{1\,mp}\,\gamma^p\eta_- - \frac{i}{8}e^\phi(e^{-i\not{J}}\not{F}_{B1})_m\eta_+ +&\\ +\frac{1}{8}e^\phi\left[-(\not{F}_{B1\,m}\not{\Omega})_p + \frac{1}{2}(\not{F}_{B1}\not{\Omega})_0 g_{mp} + (\not{F}_{B1}\not{\Omega})_{mp}\right]\gamma^p\eta_- &= 0\,,\end{aligned} \tag{A.19}$$

$$\begin{aligned}\left[2i\alpha P^{mn}q_{mn} + \frac{i}{24}\beta(\not{H}\Omega)_0\right]\eta_- +&\\ +\left[i\alpha q_m + \frac{i}{2}\alpha q_{nr}\overline{\Omega}^{nr}{}_m - \frac{1}{48}\beta(\not{H}e^{-i\not{J}})_m + \alpha\partial_m(2A - \phi + \ln\alpha)\right]\gamma^m\eta_+ &= 0\,.\end{aligned} \tag{A.20}$$

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In deriving the equations above we used the following identities:

$$
\begin{aligned}
\eta_-^\dagger \not{F}_A \eta_- &= \frac{1}{2}(\not{F}_A e^{i\not{J}})_0 \,, \\
\eta_+^\dagger \gamma_m \not{F}_A \eta_- &= -\frac{i}{2}(\not{F}_A \not{\overline{\Omega}})_m \,, \\
\eta_+^\dagger \not{F}_A \gamma_m \eta_- &= -\frac{i}{2}(\not{\overline{\Omega}} \not{F}_A)_m \,, \\
\eta_-^\dagger \gamma_p \not{F}_A \gamma_m \eta_- &= -(\not{F}_{A\,m} e^{i\not{J}})_p + \frac{1}{2}(\not{F}_A e^{i\not{J}})_0\, g_{mp} + (\not{F}_A e^{i\not{J}})_{mp}
\end{aligned} \tag{A.21}
$$

for IIA, and for IIB

$$
\begin{aligned}
\eta_-^\dagger \not{F}_B \eta_+ &= -\frac{i}{2}(\not{F}_B \not{\Omega})_0 \,, \\
\eta_+^\dagger \gamma_m \not{F}_B \eta_+ &= \frac{1}{2}(\not{F}_B e^{-i\not{J}})_m \,, \\
\eta_+^\dagger \not{F}_B \gamma_m \eta_+ &= \frac{1}{2}(e^{-i\not{J}} \not{F}_B)_m \,, \\
\eta_-^\dagger \gamma_p \not{F}_B \gamma_m \eta_+ &= -i(\not{F}_{B\,m} \not{\Omega})_p + \frac{i}{2}(\not{F}_B \not{\Omega})_0\, g_{mp} + i(\not{F}_B \not{\Omega})_{mp} \,.
\end{aligned} \tag{A.22}
$$

The explicit expressions for the Clifford products appearing in (A.15)–(A.20) are:

$$
\begin{aligned}
(\not{F}_A e^{i\not{J}})_0 &= F_0 - \frac{i}{2} F^{ab} J_{ab} - \frac{1}{8} F^{abcd} J_{ab} J_{cd} + \frac{i}{48} F^{abcdef} J_{ab} J_{cd} J_{ef} \,, \\
(\not{F}_A \not{\overline{\Omega}})_m &= -\frac{1}{2} F^{ab} \overline{\Omega}_{abm} + \frac{1}{6} F^{abc}{}_m \overline{\Omega}_{abc} \,, \\
(\not{\overline{\Omega}} \not{F}_A)_m &= -\frac{1}{2} F^{ab} \overline{\Omega}_{abm} - \frac{1}{6} F^{abc}{}_m \overline{\Omega}_{abc} \,, \\
(\not{F}_{A\,m} e^{i\not{J}})_n &= 2F_{mp} P_n{}^p + i J_{ab} F^{ab}{}_{mp} \overline{P}_n{}^p - \frac{1}{4} J_{ab} J_{cd} F^{abcd}{}_{mp} P_n{}^p \,, \\
(\not{F}_A e^{i\not{J}})_{mn} &= \frac{i}{2} F_0 J_{mn} + \frac{1}{2} F_{mn} + i F^a{}_{[m} J_{n]a} + \frac{1}{4} F^{ab} J_{ab} J_{mn} - \frac{1}{2} F^{ab} J_{am} J_{bn} - \\
&\quad - \frac{i}{4} F_{mn}{}^{ab} J_{ab} - \frac{1}{2} F^{abc}{}_{[m} J_{n]a} J_{bc} - \frac{i}{16} F^{abcd} J_{ab} J_{cd} J_{mn} + \\
&\quad + \frac{i}{4} F^{abcd} J_{ab} J_{cm} J_{dn} + \frac{1}{16} F^{abcd}{}_{mn} J_{ab} J_{cd}
\end{aligned} \tag{A.23}
$$

for IIA, where F_A is a generic sum of fluxes, i.e. $F_A = F_{(0)} + F_{(2)} + F_{(4)} + F_{(6)}$, and

$$
\begin{aligned}
(\not{F}_B \not{\Omega})_0 &= -\frac{1}{6} F^{abc} \Omega_{abc} \,, \\
(\not{F}_B e^{-i\not{J}})_m &= 2F_n \overline{P}_m{}^n + i J_{ab} F^{ab}{}_n \overline{P}_m{}^n - \frac{1}{4} J_{ab} J_{cd} F^{abcd}{}_n \overline{P}_m{}^n \,, \\
(e^{-i\not{J}} \not{F}_B)_m &= 2F_n P_m{}^n + i J_{ab} F^{ab}{}_n P_m{}^n - \frac{1}{4} J_{ab} J_{cd} F^{abcd}{}_n P_m{}^n \,, \\
(\not{F}_{B\,m} \not{\Omega})_n &= -\frac{1}{2} F^{ab}{}_m \Omega_{nab} - \frac{1}{6} F^{abc}{}_{mn} \Omega_{abc} \,, \\
(\not{F}_B \not{\Omega})_{mn} &= \frac{1}{2} F^a \Omega_{amn} - \frac{1}{2} F^{ab}{}_{[m} \Omega_{n]ab} - \frac{1}{12} F^{abc}{}_{mn} \Omega_{abc}
\end{aligned} \tag{A.24}
$$

for IIB, where $F_B = F_{(1)} + F_{(3)} + F_{(5)}$. Finally

$$
\begin{aligned}
(\not{H}\not{\Omega})_0 &= -\frac{1}{6} H^{abc}\Omega_{abc} \,, \\
(\not{H} e^{-i\not{J}})_m &= i J_{ab} H_n{}^{ab} \overline{P}_m{}^n
\end{aligned}
\tag{A.25}
$$

for both IIA and IIB.

In the text we have used the decomposition of the fluxes in SU(3) representations. These are defined in the following way: for the H-flux

$$
H = -\frac{3}{2}\,\mathrm{Im}\left(H^{(1)}\bar{\Omega}\right) + H^{(3)} \wedge J + H^{(6)} \,,
\tag{A.26}
$$

and similarly for the IIB RR 3-form flux. The components are explicitly given by

$$
\begin{aligned}
H^{(1)} &= -\frac{i}{36} H^{ijk}\Omega_{ijk} \,, \\
H^{(3)}_i &= \frac{1}{4} H_{imn} J^{mn} \,, \\
H^{(6)}_{ij} &= H^{kl}{}_{(i}\Omega_{j)kl} \,.
\end{aligned}
\tag{A.27}
$$

RR three-form flux F_3 decomposes exactly in the same way as H. For the rest of the RR fluxes we use the following decompositions

$$
\begin{aligned}
F_2 &= \frac{1}{3} F^{(1)}_2\, J + \mathrm{Re}\left(F^{(3)}_2 \llcorner \overline{\Omega}\right) + F^{(8)}_2 \,, \\
F_4 &= \frac{1}{6} F^{(1)}_4 J \wedge J + \mathrm{Re}\left(F^{(3)}_4 \wedge \overline{\Omega}\right) + F^{(8)}_4 \,, \\
F_5 &= F^{(3)}_5 \wedge J \wedge J \,, \\
F_6 &= \frac{1}{6} F^{(1)}_6\, J \wedge J \wedge J \,,
\end{aligned}
\tag{A.28}
$$

where the different representations are given by

$$
\begin{aligned}
F^{(1)}_2 &= \frac{1}{2} F_{mn} J^{mn} = F_{i\bar{j}} J^{i\bar{j}} \,, \\
F^{(3)}_{2\,k} &= \frac{1}{8} F^{ij}\Omega_{ijk} \,, \\
F^{(1)}_4 &= \frac{1}{8} F^{mnpq} J_{mn} J_{pq} \,, \\
F^{(3)}_{4\,k} &= \frac{1}{24} F_k{}^{ijl}\Omega_{ijl} \,, \\
F^{(3)}_{5\,i} &= \frac{1}{16} F_i^{mnpq} J_{mn} J_{pq} \,, \\
F^{(1)}_6 &= \frac{1}{48} F^{mnpqrs} J_{mn} J_{np} J_{qr} \,.
\end{aligned}
\tag{A.29}
$$

With the definitions above, we can write the matrices Q, R and S in terms of SU(3) representations. For IIA we have

$$
\begin{aligned}
A_{\bar\imath} &= \alpha\partial_{\bar\imath}A\,,\\
S &= -\frac{i}{4}e^{\phi}\left[\beta^*F_0 - i\alpha^*F_2^{(1)} - \beta^*F_4^{(1)} + i\alpha^*F_6^{(1)}\right]\,,\\
S_{\bar\imath} &= \frac{1}{2}e^{\phi}(\alpha^*F_2^{(\bar3)} + \beta^*F_4^{(\bar3)})_{\bar\imath}\,,\\
Q_{\bar\imath} &= -i\partial_{\bar\imath}\alpha - \frac{i}{2}\Big[\alpha(W_5 - W_4) + i\beta H^{(3)}\Big]_{\bar\imath}\,,\\
Q_{ij} &= -\frac{1}{8}\Big[\Omega_{ijk}(\alpha W_4 + i\beta H^{(3)})^k + \frac{i}{2}(\alpha W_3 + i\beta H^{(6)})_i{}^{kl}\Omega_{jkl}\Big]\,,\\
Q_{\bar\imath j} &= \frac{1}{4}\Big[(\alpha W_1 - 3i\beta H^{(1)})g_{\bar\imath j} + i\alpha W_{2\,\bar\imath j}\Big]\,,\\
R_i &= 0\,,\\
R_{\bar\imath} &= \frac{i}{2}e^{\phi}(\alpha^*F_2^{(\bar3)} - \beta^*F_4^{(\bar3)})_{\bar\imath}\,,\\
R_{ij} &= 0\,,\\
R_{\bar\imath j} &= -\frac{1}{8}e^{\phi}\left[g_{\bar\imath j}\bar S - g_{\bar\imath j}(\frac{8}{3}\beta F_4^{(1)} - \frac{8}{3}\alpha^*F_6^{(1)}) - 2\alpha^*F^{(8)}_{2\,\bar\imath j} - 2i\beta F^{(8)}_{\bar\imath j\bar k l}J^{\bar k l}\right]\,,\\
T &= \frac{3}{4}(\alpha i W_1 - \beta H^{(1)})\,,\\
T_{\bar\imath} &= \alpha\partial_{\bar\imath}(2A - \phi - \log\alpha) + \frac{1}{2}\Big[\alpha(W_4 + W_5) - i\beta H^{(3)}\Big]_{\bar\imath}\,.
\end{aligned}
\tag{A.30}
$$

For IIB we get

$$
\begin{aligned}
A_{\bar\imath} &= \alpha\partial_{\bar\imath}A\,,\\
S &= \frac{3}{2}i\beta e^{\phi}F_3^{(1)}\,,\\
S_i &= \frac{1}{4}e^{\phi}\Big(\alpha^*F_1^{(3)} + 2i\beta^*F_3^{(3)} - 2\alpha^*F_5^{(3)}\Big)_i\,,\\
S_{\bar\imath} &= \frac{1}{4}e^{\phi}\Big(\alpha F_1^{(\bar3)} - 2i\beta F_3^{(\bar3)} - 2\alpha F_5^{(\bar3)}\Big)_{\bar\imath}\,,\\
Q_{\bar\imath} &= -i\partial_{\bar\imath}\alpha - \frac{i}{2}\Big[\alpha(W_5 - W_4) - i\beta H^{(3)}\Big]_{\bar\imath}\,,\\
Q_{\bar\imath j} &= \frac{1}{4}\Big[(\alpha W_1 + 3i\beta H^{(1)})g_{\bar\imath j} + i\alpha W_{2\,\bar\imath j}\Big]\,,\\
Q_{ij} &= -\frac{1}{8}\Big[\Omega_{ijk}(\alpha W_4 - i\beta H^{(\bar3)})^k + \frac{i}{2}(\alpha W_3 - i\beta H^{(6)})_i{}^{kl}\Omega_{jkl}\Big]\,,\\
R_i &= -\frac{i}{4}e^{\phi}\Big(\alpha F_1^{(3)} - 2i\beta F_3^{(3)} - 2\alpha F_5^{(3)}\Big)_i\,,\\
R_{\bar\imath} &= 0\,,\\
R_{ij} &= -\frac{i}{16}e^{\phi}\Big(\alpha F_1^{(\bar3)k}\Omega_{ijk} - \beta F^{(6)}_{3ij} + 2\alpha F_5^{(\bar3)k}\Omega_{ijk}\Big)\,,\\
R_{\bar\imath j} &= 0\,,\\
T &= \frac{3}{4}(\alpha i W_1 + \beta H^{(1)})\,,\\
T_{\bar\imath} &= \alpha\partial_{\bar\imath}(2A - \phi - \log\alpha) + \frac{1}{2}\Big[\alpha(W_4 + W_5) + i\beta H^{(3)}\Big]_{\bar\imath}\,.
\end{aligned}
\tag{A.31}
$$

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